Lecture 2: Introduction to bond percolation and critical probability

9th August, 13th August and 16th August 2007

2.1 Bond percolation: Notation and definitions

We now turn our attention to bond percolation in d-dimensional lattices. Let us begin by introducing some notation and defining our terms.

Our basic structure is a graph that we call *d-dimensional lattice*, denoted $\mathbb{L}^d$. Each vertex in this graph is a d-dimensional vector of integers i.e. the vertex set is $\mathbb{Z}^d$. We denote the edge set by $\mathbb{E}^d$ and it is defined as follows:

$$\mathbb{E}^d = \{(u,v)|u,v \in \mathbb{E}^d, \sum_{i=1}^{d} |u_i - v_i| = 1\},$$

where u_i is the ith component of the vertex u.

In the bond percolation setting each edge $e \in \mathbb{E}^d$ has two states, *open* and *closed*. Each edge is in an open state with probability p for some $0 \leq p \leq 1$ and closed with probability $q = 1 - p$. Formally speaking we take a probability space in which the sample space is $\Omega = \prod_{e \in \mathbb{E}^d}\{0,1\}$, points of which are denoted by $\omega = (\omega(e) : e \in \mathbb{E}^d)$ and are referred to as *configurations*. The value $\omega(e) = 1$ corresponds to the edge e being open and $\omega(e) = 0$ corresponds to it being closed. The σ-field of this probability space, $\mathcal{F}$, is the subsets of Ω generated by the finite dimensional cylinders i.e. each element of $\mathcal{F}$ corresponds to a subset of the configurations in Ω in which the state of the edges in some finite subset of $\mathbb{E}^d$ is fixed.

The measure defined on this space is a product measure

$$P_p = \prod_{e \in E_d} \mu_e,$$

where $\mu(e)$ is a Bernoulli measure on $\{0,1\}$ for each e, given by

$$\mu_e(\omega(e) = 1) = p \text{ and } \mu_e(\omega(e) = 0) = 1 - p.$$

For example, the probability of a configuration having $e_1 = 1$, $e_2 = 0$, $e_3 = 1$ is $\mu_{e_1}(e_1 = 1) \cdot \mu_{e_2}(e_2 = 0) \cdot \mu_{e_1}(e_1 = 1) = p^2(1-p)$.

A natural partial order can be defined on Ω. Informally we say that for $\omega_1, \omega_2 \in \Omega$, $\omega_1 \leq \omega_2$ if an edge that is open in w_1 is also open in w_2. Formally we say that $omega_1 \leq \omega_2$ if $\forall e : \omega_1(e) \leq \omega_2(e)$. Let us define

$$K(\omega) = \{e \in \mathbb{E}^d | \omega(e) = 1\},$$

i.e. $K(\omega)$ is collection of all edges open in configuration ω. We can see that the partial order defined on Ω corresponds to the subset relation for the sets $K(\omega)$ i.e.

$$K(\omega_1) \subseteq K(\omega_2) \Leftrightarrow \omega_1 \leq \omega_2.$$

An *open cluster* (respectively *closed cluster*) is a connected component in the subgraph of $\mathbb{L}^d$ which uses only the open (respectively closed) edges of $\mathbb{E}^d$. By default when we say cluster, we mean the open cluster i.e. a connected component made up of the open edges in $\mathbb{L}^d$.

For a vertex $x \in \mathbb{Z}^d$, we define $C(x)$ to be the open cluster containing x. Since we often consider the open cluster containing the origin, we will simply write C for that cluster and mention x only when we are talking about a cluster containing some other vertex. Note that there is a probability distribution over the different possibilities for $C(x)$. This probability distribution has an important property that is used often in percolation: The probability distribution of $C(x)$ is independent of x. Formally, this is due to the *translation invariance* of the lattice and of the probability measure defined on it (c.f. [1]). Intuitively, we can think of the symmetry of any two vertices in the percolation setting.

Let us consider a very simple example of another such translation invariant property. Consider the family of events $A_x : C(x) = \{x\}$ i.e. the event that the cluster containing x contains only x. It's easy to see that $\mathrm{P}(A_x) = (1-p)^{2d}$ no matter what x is.

Similarly the probability that a vertex and all its neighbours lie in the same connected component is the same no matter which vertex we consider. However the probability that a given vertex v lies in the open component containing x depends on that x we choose i.e. it is *not* a translation invariant property..

Let A be a set of vertices. The *surface* of A, denoted $\partial(A)$ is

$$\partial(A) = \{x \in A | \exists y \notin A : (x, y) \in \mathbb{E}^d\}.$$

There are certain groups of vertices that will be referred to often. One of them is a *box* defined as

$$B(a, b) = \{x \in Z_d | \forall i a_i \leq x_i \leq b_i\}$$

A special kind of box is the box of side length $2n$ centred at the origin: $B(n) = B(n, -n)$.

2.2 Critical Probability

The *percolation probability*, denoted $\theta(p)$ is a key figure in the study of percolation. It is defined as

$$\theta(p) = P_p(|C| = \infty)$$

A fundamental fact of percolation theory is that there is a critical value $p_c(d)$ (often written just p_c when the dimension is understood) such that

$$\theta(p) = \begin{cases} 0 & \text{if } p < p_c \\ 1 & \text{if } p > p_c \end{cases}$$

This value, $p_c(d)$ is called the critical probability and is defined as

$$p_c(d) = \sup\{p | \theta(p) = 0\}.$$

Let us consider the case of $d = 1$. Here we claim that p_c has the trivial value 1.

Claim 2.1

$$p_c(1) = 1.$$

Proof. For $i \geq 0$ define A_i to be the event that all edges starting from node 2^k to $2^{k+1} - 1$ are open and edges from $-2^{k+1} + 1$ to -2^k are open. Note that

$$\mathrm{P}(A_k) = p^{2^k} \cdot p^{2^k}.$$

Hence, as long as $p < 1$. $\sum_{k=1}^{\infty} P(A_k) < \infty$. And so, by the Borel-Cantelli lemma we get that $P(A_k \text{ i.o.}) = 0$ i.e. there must be some k for which A_k is not true with probability 1. And this implies that as long as $p < 1$, the probability of having an infinite component is 0. $\blacksquare$.

However for $d \geq 2$ we get non-trivial critical probabilites. The rest of this lecture is devoted to establishing this. We begin by claiming that the critical probability is a non-increasing function of the dimension.

Claim 2.2 *For* $d \geq 1$,

$$p_c(d+1) \leq p_c(d).$$

This claim follows from observing that at any value of p, if you have a infinite component in $\mathbb{L}^d$ then you will also have an infinite component in $\mathbb{L}^{d+1}$. We claim, without the proof that a strict version of this inequality also holds.

With this in view, we now proceed to show that the 2-dimensional mesh has a non-trivial critical probability.

Theorem 2.3

$$\frac{1}{3} \leq p_c(2) \leq \frac{2}{3}.$$

Proof.

Let us first show that $p_c(2) > 1/3$. Denote by $\sigma(n)$ the number of paths of length n in $\mathbb{L}^2$ beginning at the origin. We define a random variable $N(n)$ to be the number of these paths that are open. Now, if the origin is a part of an infinite cluster then there exist paths of all possible lengths starting at the origin. Restating this in terms of $N(n)$ we can that the event that the origin is part of an infinite cluster implies the event that $N(n)$ is at least 1 for each value of n i.e. for all n

$$\begin{aligned} \theta(p) &\leq \mathrm{P}_p(N(n) \geq 1) && (1) \\ &\leq \mathrm{E}_p(N(n)) && (2) \end{aligned}$$

Now, the probability of a given path of length n being open is p^n and there are $\sigma(n)$ such paths, so

$$E_p(N(n)) \quad = \quad p^n \sigma(n). \tag{3}$$

Let us now try to bound $\sigma(n)$. Every path starts at the origin so there are four choices for the first edge. After that at each vertex on the path we have at most 3 choices, eliminating the obviously wrong choice of going back to the vertex we just came from. Hence

$$\sigma(n) \leq 4 \cdot 3^{n-1}.$$

Using this and (3) in (1-2) we get

$$\theta(p) \leq \frac{4}{3}(3p)^n.$$

Hence, if $3p < 1$, $\theta(p) \to 0$ as $n \to \infty$, proving that for $p < 1/3$, the origin cannot be part of an infinite cluster, hence p_c must be at least $1/3$.

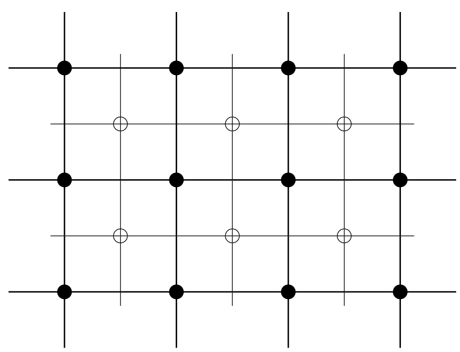

Figure 1: The dual of $\mathbb{L}^2$.

In order to prove the second inequality i.e $p_c \leq 2/3$, we will introduce the *dual* of the lattice $\mathbb{L}^2$ which we will denote $\mathbb{L}^2_d$. This is defined in the way that the dual of a planar graph is usually defined in graph theory, by placing a vertex in every face and by putting an edge between two vertices corresponding to two faces that share an edge (see Figure 1.) Note that there is a natural one-to-one correspondence between the edges of the dual and the edges of $\mathbb{L}^2$. We define a bond percolation process on the dual using this correspondence by declaring an edge of the dual to be closed if it crosses a closed edge of $\mathbb{L}^2$ and open if the edge of $\mathbb{L}^2$ it crosses is open.

By convention we denote the vertices of the dual as $(x + \frac{1}{2}, y + \frac{1}{2})$ where x and y are integers.

Now, to see the line our proof will take, let us note that if the origin was contained in a finite cluster that would mean there is a set of closed edges surrounding this cluster. This set of edges corresponds to a circuit of closed edges surrounding the origin in the dual. With this in mind we can proceed.

First, let us count the number of circuits of length n surrounding the origin in the dual. We denote this quantity $\gamma(n)$. Note that any such circuit must touch on a point of the form $(k+\frac{1}{2}, \frac{1}{2})$ for some $0 \leq k \leq n$. We can now consider each circuit as a walk of length $n-1$ beginning from such a vertex and returning to some neighbour of $(k + \frac{1}{2}, \frac{1}{2})$. Hence, given the number of choices for k, and recalling the definition of $\sigma(n)$ from the earlier part of the proof, we have

$$\gamma(n) \leq n\sigma(n-1) \leq \frac{4}{3} \cdot n \cdot 3^{n-1}.$$

Let us denote by $M(n)$ the number of these circuits of the dual of length n surrounding the origin which are closed. Since a circuit of length n is closed with probability $(1-p)^n$ we get that

$$\mathrm{E}_p(M(n)) \leq \frac{4}{3} \cdot n \cdot (3(1-p))^{n-1}. \tag{4}$$

Now, define two events G_m and F_m as follows

- G_m : All edges in $B(m)$ are open.
- F_m : There is a closed circuit in $\mathbb{L}^2_d$ containing $B(m)$ in its interior.

Note that if G_m occurs and F_m occurs, then there is a path of length n starting at the origin for all values of n. Hence the event $G_m \wedge \bar{F_m}$ implies the event that the origin is part of an infinite component i.e.

$$\theta(p) \geq \mathrm{P}_p(G_m \wedge \bar{F_m}) = \mathrm{P}_p(G_m) \cdot \mathrm{P}_p(\bar{F_m}). \tag{5}$$

The latter part following from the fact that G_m and F_m are events depending on disjoint sets of edges.

Let us now try to upper bound $\mathrm{P}_p(F_m)$ so that we can complete the lower bound in 5.

We know that a circuit surrounding $B(m)$ must have length at least as much as the perimeter of $B(m)$ i.e. $8m$. So, for F_m to be true there must be a circuit of length at least $8m$ surrounding the origin that is closed. Note that all circuits of this length may not contain $B(m)$ completely, and hence we get the following inequality

$$\mathrm{P}_p(F_m) \leq \mathrm{P}_p\left(\sum_{n=8m}^{\infty} M(n) \geq 1\right)$$

Now, since $\mathrm{P}_p(M(n) \geq 1) \leq \mathrm{E}_p(M(n))$, using 4, we have

$$\mathrm{P}_p(F_m) \leq \sum_{n=8m}^{\infty} \frac{4}{3} \cdot n \cdot (3(1-p))^{n-1}.$$

If $3(1-p) < 1$ this sum converges, and it is possible to find a large enough value of m^* of m such that $\mathrm{P}_p(F_{m^*}) \leq \frac{1}{2}$. Using this in 5, and noting the fact that since m is a finite value $\mathrm{P}_p(G_m) > 0$ for any $p > 0$, we get

$$\theta(p) \geq \frac{1}{2} \cdot \mathrm{P}_p(G_{m^*}) > 0.$$

Hence, if $p > 2/3$ we are definitely above the critical probability. And this completes the proof of the second inequality. ■

References

[1] http://en.wikipedia.org/wiki/Translational_symmetry.