Lecture 10: Supercritical phase: Exponential decay of the radius of a finite open cluster

25th, 29th October and 1st November 2007

In the subcritical phase we saw that the radius of the cluster containing the origin decays exponentially. This is not true if $p > p_c$ since there is a non-zero probability that the origin may be part of an infinite open cluster. However, even in the supercritical phase we expect to find finite open clusters along with the infinite open cluster. In this lecture we will show that the radius of these finite clusters decays exponentially.

We will proceed in two stages. First, we will show that for $0 < p < 1$, there is a $\sigma(p) < \infty$ such that $\mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty)$ is upperbounded by a quantity which contains the term $e^{-n\sigma(p)}$, where C is, as before, the open cluster containing the origin. In the second stage we will show that for $p > p_c$, this $\sigma(p)$ is strictly positive, thereby proving an exponential decay in the probability.

10.1 An upper bound for all values of p.

Theorem 10.1 *Suppose that* $0 < p < 1$. *The limit*

$$\sigma(p) = \lim_{n\to\infty} \left\{ \frac{-1}{n} \log P_p(0 \leftrightarrow \partial B(n) \cap |c| < \infty) \right\}$$

exists and satisfies $\sigma(p) < \infty$. *Furthermore, there exists a constant* $A(p, d)$ *which is finite for* $d \geq 2$, $0 < p < 1$, *such that*

$$P_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty) \leq A(p, d) n^d e^{-n\sigma(p)} \tag{1}$$

for all $n \geq 0$.

Proof. Let us begin with some notation. For each dimension $i = 1, \ldots, d$, we denote the maximum and the minimum values reached by the vertices of C in that dimension by $R_i = \max\{x_i : x \in C\}$ and $L_i = \min\{x_i : x \in C\}$. $D_i = R_i - L_i$ is the *width* of C in the ith coordinate. And we define the *diameter* of C, denoted $\text{diam}(C)$ as the maximum over the widths in the different dimensions i.e.

$$\text{diam}(C) = \max\{D_i : 1 \leq i \leq d\}.$$

Our proof will involve writing out a recurrence relating the probability that the diameter is at least some value to the product of the probabilities of the diameter being some smaller values. Once we have this recurrence, we will be able to take logs and apply the subadditive limit theorem. Let us first prove this recurrence.

Lemma 10.2 *Suppose* $0 < p < 1$. *For* $m, n \geq 0$

$$P_p(\mathit{diam}(C) = m+n+2) \geq \frac{p^2(1-p)^{2d-2}}{d^2(2n+1)^d} \cdot P_p(\mathit{diam}(C) = m) \cdot P_p(\mathit{diam}(C) = n). \tag{2}$$

Proof of Lemma 10.2. In order to lower bound $\text{P}_p(\text{diam}(C) = m+n+2)$ we will focus on a particular way of constructing a cluster of diameter $m+n+2$: We will place a rightmost vertex of an open cluster with $D_i = m$ two hops away from a leftmost vertex of an open cluster with $D_i = n$, then we will open the two edges separating these two clusters. The probability of such a structure existing will be our lower bound.

Let us begin by noting that

$$\text{P}_p(\text{diam}(C) = k) \geq \text{P}_p(D_1 = \text{diam}(C) = k). \tag{3}$$

Also, since all the dimensions are symmetric and at least one of them must have width equal to the diameter

$$\begin{aligned} \text{P}_p(\text{diam}(C) = k) &\leq \sum_{i=1}^{d} \text{P}_p(D_i = \text{diam}(C) = k) \\ &\leq d \cdot \text{P}_p(D_1 = \text{diam}(C) = k). \end{aligned} \tag{4}$$

Now, note that if $D_1 = \text{diam}(C) = k$, then C must be contained in $B(k)$, since the origin is in C. This means that any leftmost vertex of C in the 1st dimension must also belong to $B(k)$. Hence

$$\begin{aligned} &\text{P}_p(\text{A leftmost vertex of } C \text{ is contained in } B(k) \cap D_1 = \text{diam}(C) = k) \\ &\qquad = \text{P}_p(D_1 = \text{diam}(C) = k). \end{aligned}$$

This means there must be a vertex x^* in $B(k)$ such that

$$\begin{aligned}\mathrm{P}_p(x^* \text{ is a leftmost vertex of } C\ \cap D_1 = \mathrm{diam}(C) = k) \\ \geq \frac{1}{|B(k)|}\mathrm{P}_p(D_1 = \mathrm{diam}(C) = k).\end{aligned}$$

The vertex x^* is, therefore, a leftmost vertex of an open cluster with diameter k whose width is maximal in the first coordinate. Since this property is translation invariant, the probability that the origin has this property is the same as the probability that x^* has this property. So, we can say that

$$\mathrm{P}_p(D_1 = \mathrm{diam}(C) = k \cap L_1 = 0 \cap R_1 = k) \geq \frac{1}{|B(K)|}\mathrm{P}_p(D_1 = \mathrm{diam}(C) = k)$$

And further using (4), we get

$$\mathrm{P}_p(D_1 = \mathrm{diam}(C) = k \cap L_1 = 0 \cap R_1 = k) \geq \frac{1}{d|B(k)|}\mathrm{P}_p(\mathrm{diam}(C) = k). \quad (5)$$

Now, let us pick a rightmost vertex of C, breaking ties in some predetermined way. Let this rightmost vertex be x. Consider the event that $x + (2, 0, \ldots)$ is

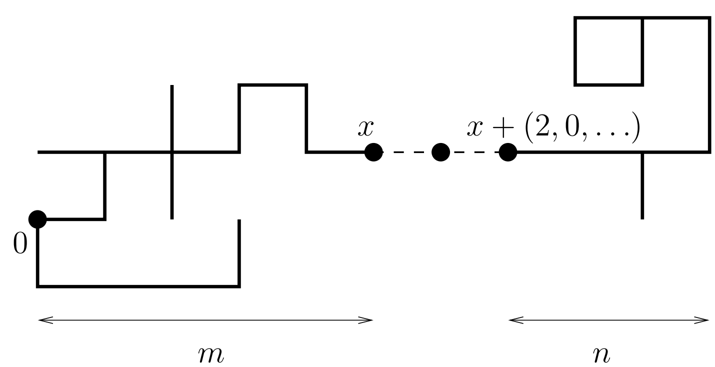

Figure 1: Juxtaposing two open clusters of width m and n.

a leftmost vertex of the open cluster containing it, and that this open cluster has diameter n and its width is maximal in the first coordinate. Figure 1 illustrates this event. If we now close the $2d-2$ edges incident to $x+(1, 0, \ldots)$ and open the two edges along the 1st coordinate, we get a cluster of diameter $m + n + 2$, with maximum width only along the 1st dimension.

So, in order to lower bound the probability of an open cluster having diameter $m + n + 2$ we consider the co-occurence of three events:

- the cluster containing the origin has width m in the first coordinate,
- the vertex $x + (2, 0, \ldots)$, two hops away from a rightmost vertex x of C, is a leftmost vertex of the cluster $C(x + (2, 0, \ldots))$ containing it, and $\mathrm{diam}(C(x + (2, 0, \ldots))) = n$ with maximum width in the 1st coordinate and
- the edges from x to $x + (1, 0, \ldots)$ and $x + (1, 0, \ldots)$ to $x + (2, 0, \ldots)$ are open and all other edges incident to $x + (1, 0, \ldots)$ are closed.

The inequality (3) gives us a way of lower bounding the probability of the first of these three events. the inequality (5) allows us to lowerbound the probability of the second event, using translation invariance to apply it to $x + (2, 0, \ldots)$ rather than the origin. The probability for the third event we can write down explicitly. Note also that these three events pertain to three disjoint sets of edges. Hence we can put all of this together to say

$$\begin{aligned} \mathrm{P}_p(\mathrm{diam}(C) = m + n + 2) \geq \frac{1}{d}\mathrm{P}_p(\mathrm{diam}(C) = m) \cdot p^2(1-p)^{2d-2} \\ \cdot \frac{1}{d|B(n)|} \cdot \mathrm{P}_p(\mathrm{diam}(C) = n). \end{aligned}$$

And since $|B(n)| = (2n + 1)^d$ the lemma follows. ■

Now, define a sequence $\delta(k) = -\log P_p(\mathrm{diam}(C) = k)$. Taking logs on both sides of (2) we get

$$\delta(m + n + 2) \leq \delta(m) + \delta(n) + \log \left\{ \frac{d^2(2n + 1)^d}{p^2(1 - p)^(2d - 2)} \right\}. \tag{6}$$

Applying a variant of the Subadditive Limit Theorem (described in Section 10.3), we get that since the third term on the righthand side of (6) grows asymptotically slower than n, the limit

$$\sigma(p) = \lim_{n\to\infty} \left\{ \frac{\delta(n)}{n} \right\}$$

exists and

$$\delta(k) \geq (k + 2)\sigma(p) - \log \left\{ \frac{d^2(2k + 1)^d}{p^2(1 - p)^(2d - 2)} \right\}.$$

This gives us

$$\mathrm{P}_p(\mathrm{diam}(C) = k) \leq \frac{d^2(2k + 1)^d}{p^2(1 - p)^(2d - 2)} \cdot e^{-(k+2)\sigma(p)}. \tag{7}$$

Note now that if the origin is connected to $\partial B(n)$ then the diameter of C must be at least n and if $|C|$ is finite then the diameter of C must be finite. Hence

$$\mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty) \leq \mathrm{P}_p(n \leq \mathrm{diam(C)} < \infty).$$

Using (7) we get

$$\begin{aligned} P_p(0 \leftrightarrow \partial B(n) \cap |\mathrm{C}| < \infty) &\leq \sum_{k=n}^{\infty} \frac{d^2(2k+1)^d}{p^2(1-p)^(2d-2)} \cdot e^{-(k+2)\sigma(p)} \\ &\leq \frac{d^2(2n+1)^d}{p^2(1-p)^(2d-2)} \cdot e^{-(n+2)\sigma(p)} \cdot \gamma \end{aligned} \tag{8}$$

where γ is some finite quantity. This satisfies the requirement (1) of the statement of the theorem. Further, taking logs on both sides we get

$$-\frac{1}{n}\log(\mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty)) \geq \sigma(p) + \frac{1}{n}\gamma' \tag{9}$$

where γ' is some finite quantity independent of n.

Also if the origin is a leftmost vertex of a cluster with diam $= n$ which has width n in the 1st coordinate, then $\{0 \leftrightarrow \partial B(n)\}$ and the finiteness of $|C|$ are both implied. So

$$P_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty) \geq P_p(\mathrm{diam}(C) = n \cap L_1 = 0 \cap R_1 = n).$$

Using (5) we get

$$\mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty) \geq \frac{1}{d(2n+1)^d}\mathrm{P}_p(\mathrm{diam}(C) = n).$$

Taking logs on both sides we get

$$-\frac{1}{n}\log(\mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty))$$

Now, using (8) we get

$$\begin{aligned} -\frac{1}{n}\log(\mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty)) \leq \frac{\log d}{n} + \frac{d\log(2n+1)}{n} \\ -\frac{1}{n}\log(\mathrm{P}_p(\mathrm{diam}(C) = n)) \end{aligned}$$

Recalling that

$$\sigma(p) = \lim_{n\to\infty} \left\{ \frac{-\log \mathrm{P}_p(\mathrm{diam}(C) = n)}{n} \right\}$$

we get that as $n \to \infty$

$$-\frac{1}{n} \log(\mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty)) \leq \sigma(p). \tag{10}$$

This along with (9) gives us the result. ∎

10.2 Decay in the supercritical phase

We now proceed to look specifically at the supercritical phase. We will show that $\sigma(p) > 0$ when $p > p_c$ and $d \geq 3$, thereby proving that in the supercritical phase the size of a finite cluster decays exponentially.

The argument depends on an important feature of percolation in higher dimensions: the critical probability of infinite slabs of $\mathbb{Z}^d$ is the same as the critical probability of $\mathbb{Z}^d$. The proof proceeds by arguing that if C has large radius but is finite, that means C intersects a large number of infinite slabs without intersecting the infinite open cluster contained in any of these slabs. And this is an event with an exponentially decreasing probability since the percolation processes within each of these slabs are independent of each other. Now, let us proceed to formalize this.

Theorem 10.3 *If $p > p_c$ then $\sigma(p) > 0$, for $d \geq 3$.*

Proof. Let $H(n)$ be the hyperplane $\{x \in \mathbb{Z}^d : x_1 = n\}$ and define an event

$$G_n : \{C \text{ is finite }\} \cap \{C \text{ intersects } H(n)\}.$$

Since B(n) has $2d$ faces

$$\mathrm{P}_p(G_n) \leq \mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty) \leq 2d \cdot \mathrm{P}_p(G_n).$$

If we show that, when $p > p_c$, there is a $\gamma(p) > 0$ such that $\mathrm{P}_p(G_n) \leq e^{-n\gamma(p)}$ for all n, then

$$\mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty) \leq 2de^{-n\gamma(p)}.$$

Which in turn would mean that

$$-\frac{1}{n}\log \mathrm{P}_p(0 \leftrightarrow \partial B(n) \cap |C| < \infty) \geq \frac{\log 2d}{n} + \gamma(p).$$

Taking limit $n \to \infty$ on both sides we get

$$\sigma(p) \geq \gamma(p)$$

which would prove the theorem. Hence, show the existence of such a $\gamma(p) > 0$ is sufficient for us. So, let us proceed to show this.

Let us denote by $R(k)$ the $d-1$-dimensional slab of width k in the first coordinate i.e.

$$R_k = \{x \in \mathbb{Z}^d : 0 \leq x_1 \leq k\}.$$

In the following, when we write only p_c it will mean $p_c(\mathbb{Z}^d)$. For everything else we will mention the structure whose critical probability we are talking about.

Since $R(k)$ is an infinite graph, it experiences a critical phenomenon as well. And it is clear, since it is fully contained in $\mathbb{Z}^d$ that $p_c(R_k) > p_c$. Since $R_k \uparrow \mathbb{Z}^d$ as $k \to \infty$, $p_c(R_k) \to p_c$. This motivates the result we state next.

Theorem 10.4 *If $p > p_c$, there is an integer k such that $p > p_c(R_k)$.*

In other words, if $p > p_c$, there may be several values of k for which $p < p_c(R_k)$ but we will eventually find a large enough value of k such that $p > p_c(R_k)$. We omit the proof of this theorem, simply stating that we will use the value of k given by this theorem in the rest of the proof i.e. a value for which the slab $R(k)$ is in the supercritical phase. Let us now view $\mathbb{Z}^d$ as a set of copies of the slab R_k. Define

$$R_k(i) = \{x \in \mathbb{Z}^d : (i-1)k \leq x_1 \leq ik\}.$$

We will argue that if G_{nk} occurs for some $n \geq 1$, then each $R_k(i)$ for $1 \leq i \leq n$ is traversed by an open path from the origin (see Figure 2) and this open path does not intersect any infinite open cluster in each such region because the origin is part of a finite cluster. Since $p > p_c$, each region contains an infinite cluster, hence the probability of avoiding such a cluster is strictly less than 1 (call it $\alpha(p,k)$). There are n such regions. So

$$\mathrm{P}_p(G_{nk}) < \alpha^n.$$

Let us now formalize this argument.

We view the cluster C as being "constructed" in the following manner:

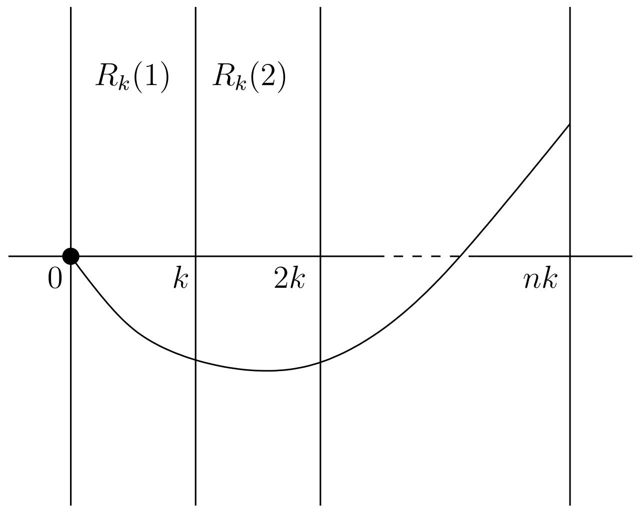

Figure 2: G_{nk} requires traversing n slabs to get to $H(nk)$.

- Order the edges of $\mathbb{E}^d$ in some arbitrary (but deterministic) way.
- Construct an increasing sequence $C_1, C_2....$ of (random) subgraphs of $\mathbb{L}^d$. C will be the limit of this sequence.
- $C_1 = 0$.
- From C_m we construct C_{m+1} by adding the earliest open edge which lies in the edge boundary of C_m, if such an edge exists. i.e. $C_{m+1} = C_m \cup \{e_j\}$ where $j = \min\{i : e_i \notin C_m, e_i$ is open and incident to a vertex of $C_m\}$.

For some m, if $C_m = C$ then we define $C_l = C$ for all $l \geq m$ so that

$$C = \lim_{m \to \infty} C_m.$$

Note that the event $\{C_m = \Sigma\}$ (for some connected subgraph Σ of $\mathbb{L}^d$) does not depend on any edge both of whose end vertices are outside Σ. Also, if $x \in C$, there is a (random) integer $m = m(x)$ such that x lies in C_m (regardless of whether C is finite or infinite).

Now, returning to the proof, construct a sequence of vertices $v_1, v_2, \ldots$ whose first element $v_1 = 0$ i.e. the origin. If m_i is the smallest value of m for which C_m contains a vertex of $R_k(i)$ then let v_i be that first vertex of

$R_k(i)$ encountered by the sequence C_m i.e. the only element of the singleton $R_k(i) \cap C_{m_i}$. So we get the seqence $v_1, v_2, \ldots, v_T$ of the "first" vertex of C in each slab, where

$$T = \sup\{i : C \cap R_k(i) \neq \emptyset\}.$$

For any region R, denote by $\theta(p, v; R)$ the probability that v belongs to an infinite open cluster of R. Recall that we chose k such that

$$p > p_c(R_k) = \sup\{p : \theta(p, 0; R_k) = 0\}$$

Hence $\theta(p, 0; R_k) > 0$. Also, let us define event $E_j(x, i)$ to be the event that vertex x does not lie in an infinite cluster of $R_j(i)$.

Now let us consider a positive integer $n = kr + s, 0 \leq s \leq k$. If G_n occurs then $T \geq r$ and v_i does not lie in an infinite cluster of $R_k(i)$ for $1 \leq i \leq r$. In other words

$$\mathrm{P}_p(G_n) \leq \mathrm{P}_p\left(\{T \geq r\} \cap \left\{\bigcap_{i=1}^{r} E_k(v_i, i)\right\}\right).$$

For brevity, let us say

$$A_j = \{T \geq r\} \cap \left\{\bigcap_{i=1}^{r} E_k(v_i, i)\right\}.$$

So, we have that $\mathrm{P}_p(G_n) \leq \mathrm{P}_p(A_r) = \mathrm{P}_p(A_r \mid A_{r-1}) \cdot \mathrm{P}_p(A_{r-1})$. So, let us turn and look at the event $\{A_j \mid A_{j-1}\}$. For the event A_j, any of the vertices of $H(j-1)k$ can be v_j, so taking the union over all these choices of v_j we can say that $\{A_j \mid A_{j-1}\}$ is

$$\bigcup_{v \in H((j-1)k)} \{v_j = v \mid A_{j-1}\} \cap \{E_k(v, j) \mid A_{j-1}\} \cap \{T \geq j \mid A_{j-1}\}.$$

Hence

$$\begin{aligned} \mathrm{P}_p(A_j|A_{j-1}) \leq & \sum_{v \in H((j-1)k)} \mathrm{P}_p(E_k(v, j) \mid v_j = v, T \geq j, A_{j-1}) \\ & \cdot \mathrm{P}_p(v_j = v \mid T \geq j, A_{j-1}) \cdot \mathrm{P}_p(T \geq j \mid A_{j-1}). \end{aligned} \tag{11}$$

Since v being in an infinite open cluster of $R_k(j)$ has nothing to do with the event A_{j-1} (by our construction of C). Also $T \geq j$ depends only on the events occuring in the slabs before the jth one, so does not affect the event $E_k(v,j)$. Hence

$$\mathrm{P}_p(E_k(v,j) \mid v_j = v, T \geq j, A_{j-1}) \leq (1 - \theta(p, v; R_k(j))).$$

We substitute this for the first term on the right side of (11) and upper bound the third term by 1 to get

$$\mathrm{P}_p(A_j \mid A_{j-1}) \leq \sum_{v \in H((j-1)k)} (1 - \theta(p, v; R_k(j))) \cdot \mathrm{P}_p(v_j = v \mid T \geq j, A_{j-1}).$$

if $T \geq j$ then one of the vertices of $H((j-1)k)$ is chosen as v_j. Additionally translating the probability $\theta(p, v; R_k(j))$ to the origin

$$\mathrm{P}_p(A_j \mid A_{j-1}) \leq (1 - \theta(p, 0; R_k(j))).$$

Using this bound we get

$$\begin{aligned} \mathrm{P}_p(G_n) &\leq \mathrm{P}_p(A_r \mid A_{r-1}) \cdot \mathrm{P}_p(A_{r-1} \mid A_{r-2}) \cdots \mathrm{P}_p(A_2 \mid A_1) \cdot \mathrm{P}_p(A_1) \\ &\leq (1 - \theta(p, 0; R_k(j)))^r. \end{aligned}$$

Since $\theta(p, 0; R_k) > 0$, we get

$$\mathrm{P}_p(G_n) \leq e^{-r\gamma(p)}$$

where

$$\gamma(p) = -\log(1 - \theta(p, 0; R_k)) > 0.$$

Since $n \leq (r+1)k$, so $r \geq \frac{n}{k} - 1$ giving

$$\mathrm{P}_p(G_n) \leq \exp\left\{-n\frac{\gamma(p)}{k} + \gamma(p)\right\}.$$

which completes the proof. ∎

10.3 Appendix: A variant of the Subadditive Limit Theorem

We state without proof a variant of the Subadditive Limit Theorem that is useful here.

Theorem 10.5 *If there is a sequence* $(x_r : r \geq 1)$ *such that for all* $m, n \geq 1$

$$x_{m+n+2} \leq x_m + x_n + g_n$$

and if $\frac{g_n}{n} \to 0$ *as* $n \to \infty$ *then*

$$\lambda = \lim \frac{x_r}{r}$$

exists, with $\lambda < \infty$ *and, for all* r

$$x_r \geq (r+2)\lambda - g_r.$$