Lecture 1: Overview of percolation and foundational results from probability theory

30th July, 2nd August and 6th August 2007

1.1 Overview

The area of Percolation theory began in 1957 [1] when Broadbent and Hammersley decided to formulate a simple stochastic model for the question:

> If a large porous stone is immersed in a bucket of water, does the centre of the stone get wet?

They looked at a porous stone as a grid of open and closed channels, each channel being open or closed with some probability. In two dimensions their porous stone can be thought of as the plane square lattice i.e. a graph in which each point of $\mathbb{Z}^2$ is a vertex and there is an edge between two nodes (x_1, y_1) and (x_2, y_2) if $|x_1 - x_2| + |y_1 - y_2| = 1$. Edges model channels in the stone. The model is parameterized with a single probability value p. Each edge (and hence each channel) is considered to be open with probability p and closed with probability $1 - p$. Furthermore we assume that each edge is open or closed independent to the state of all the other edges. This is probably a very strong assumption when it comes to stones, but it makes for an interesting and widely applicable model.

Leaving geology behind, let us now consider the infinite mesh graph, each of whose edges is open, independent of all other edges, with probability p. We will call this a *percolated lattice*. There are several questions that can be asked of this structure:

- Are there infinite-sized connected components in this percolated lattice? If so, how many of them are there?
- Does a particular vertex belong on an infinite-sized component?
- If there is no infinite-sized component, what is the size of the largest component?

- What is the distribution of the sizes of finite components?

And many, many other questions. Percolation theory tries to answer these and other questions for different values of p.

In particular if we focus on the question of the existence of an infinite component, intuition suggests that as p goes to 0, there should not be any such component and that as p tends towards 1, the whole lattice, more or less, should be a massive infinite-sized component. Therefore, in between these two extreme points, some change must occur. Suppose we consider the probability, $\theta(p)$, of an infinite component existing, the question is how does $\theta(p)$ change with p. Percolation theory answers this question: this function jumps from being 0 almost surely to being 1 almost surely. Kesten's celebrated result [2] states that the point at which this changeover happens, known as the *critical probability*, is $p = 1/2$ for the two dimensional lattice.

In this class we will work towards developing the techniques required to prove results about percolated lattices. We begin by introducing some required foundational material from probability theory.

1.2 Foundational results from probability theory

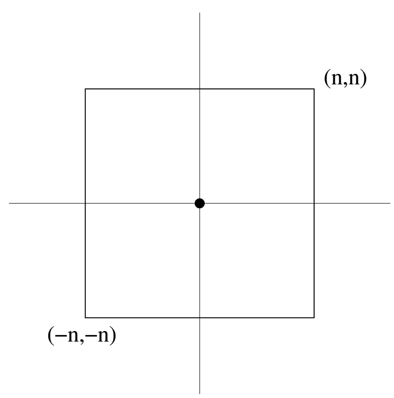

Figure 1: A square box of side length $2n$ around the origin.

Often when we talk about properties of the percolated lattice we start by looking at finite sublattices. For example, we may consider the square box of side $2n$ surrounding the origin, see Figure 1.2. We then try to deduce properties of the infinite percolated by viewing it as the limiting case of the square box as $n \to \infty$. Hence the properties of infinite sequences of events are important to us. With that in mind, we now build the discussion towards a presentation of the Borel-Cantelli Lemmas and Kolmogorov's zero-one law.

Definition 1.1 *A* probability space *is a triple* $(\Omega, \mathcal{F}, p)$, *where*

- Ω *is a set of* outcomes,
- $\mathcal{F} \subseteq 2^\Omega$ *is a set of* events *which is a* σ-field *i.e. a set of with the properties*
  1. $\emptyset$ *and* Ω *are in* $\mathcal{F}$,
  2. *if* $A \in \mathcal{F}$ *then* $A^c \in \mathcal{F}$ *i.e.* $\Omega \setminus A \in \mathcal{F}$,
  3. *for a sequence of events* $\{A_n\}$, $\cap_i A_i \in \mathcal{F}$.
- p *is a* measure *i.e. a function from* Ω *to* $[0, 1]$ *with the properties*
  1. $p(\emptyset) = 0$,
  2. $p(A) = 1 - p(A^c)$,
  3. $p(\cup_i A_i) = \sum_i p(A_i)$ *if* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$.

Let us illustrate this definition with a simple example.

1.2.1 A coin tossing example

Example 1.2 *Toss a coin with two possible outcomes,* H *and* T.

Here $\Omega = \{H, T\}$. We could have two different sets of events here: $\mathcal{F}_1 = \{\emptyset, \{H, T\}\}$ and $\mathcal{F}_2 = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$. Let us assume for now that there is some probability measure p defined on Ω. This could then be extended to define probabilities of the elements of $\mathcal{F}_1$ or $\mathcal{F}_2$.

Let us now extend this example a little.

Example 1.3 *Toss two coins, each with two possible outcomes,* H *and* T.

Here, the set of outcomes $\tilde{\Omega} = \{HH, HT, TH, TT\}$. Let us denote these outcomes $w_{i,j}$ where the two indices correspond to the two coin tosses and the value of an index is 1 if the corresponding coin toss is H and 2 if it is T.

We can also define $\tilde{\mathcal{F}} = 2^{\tilde{\Omega}}$, which is a set of 16 different events. Further we define a special kind of measure $\tilde{p}$ with the property that

$$\tilde{p}(w_{i,j}) = p(w_i) \cdot p(w_j).$$

where p is the measure defined in Example 1.2. This kind of measure, $\tilde{p}$, obtained by taking the product of the measures defined for the individual coin tosses, is called a *product measure*. A product measure can be defined for measurable spaces in general, here we state the definition for probability spaces.

Definition 1.4 *Given two probability spaces* $(\Omega_1, \mathcal{F}_1, p_1)$ *and* $(\Omega_2, \mathcal{F}_2, p_2)$*, if we consider a space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{p}$ *in which* $\tilde{\Omega} = \Omega_1 \times \Omega_2$ *and* $\tilde{\mathcal{F}}$ *consists of sets of form* $A_1 \times A_2$ *where* $A_1 \in \mathcal{F}_1$ *and* $A_2 \in \mathcal{F}_2$*, then* $\tilde{p}$ *is said to be a* product measure *if*

$$\tilde{p}(A_1 \times A_2) = p_1(A_1) \cdot p_2(A_2).$$

We now extend the example even further.

Example 1.5 *Toss an infinite number of coins.*

Here each outcome of Ω is an infinite sequence of H and T. Events are (finite or infinite) collections of these infinite sequences. Again, we can define p to be a product measure.

We can define a sequence of events $\{A_i\}_{i \geq 1}$ as follows:

$$A_n = \{(w_1, w_2, \ldots) \in \Omega | w_1 = H, w_2 = H \cdots w_n = H\}.$$

i.e. the event A_n is the set of all outcomes for which the first n coin tosses yield the result H.

Before discussing this example in greater detail we take a small detour into real sequences and define the notions of lim sup and lim inf to motivate the statement of the Borel-Cantelli lemma.

1.2.2 Bounding sequences

Given a real sequence $\{a_n\}$, it may not always happen that $\lim_{n\to\infty} a_n$ exists. So, in order to get some handle on the behaviour of the sequence we define a sequence $\{b_n\}$

$$b_n = \sup_{j\geq n} a_j$$

Note that b_n is a non-increasing sequence. So it must have a limit (which may be ∞.) We define

$$\lim_{n\to\infty} \sup_{j\geq n} a_j \stackrel{\Delta}{=} \lim_{n\to\infty} b_n.$$

Note that if we take a sequence of sets S_n, then the non-increasing sequence b_n formed from the supremums of the tails of the sequence is just a sequence of unions of the tails i.e.

$$b_n = \bigcup_{j\geq n} S_j$$

and, since b_n is a non-increasing sequence of sets, the limit of this sequence is simply the intersection of all these sets, i.e.

$$\lim_{n\to\infty} \sup_{j\geq n} S_j = \bigcap_{n\geq 1} \bigcup_{j\geq n} S_j$$

To bound real sequences from below, we have a corresponding notion.

$$\lim_{n\to\infty} \inf_{j\geq n} a_j \stackrel{\Delta}{=} \lim_{n\to\infty} \inf_{j\geq n} a_j.$$

And for sequences of sets we have

$$\lim_{n\to\infty} \inf_{j\geq n} S_j = \bigcup_{n\geq 1} \bigcap_{j\geq n} S_j$$

based on the observation that the infimum of a number of sets is their intersection.

1.2.3 Borel-Cantelli Lemmas

Given a sequence of event $\{A_n\}$ we note that the $\limsup A_n$ is also referred to a A_n *infinitely often* (or A_n i. o.) since it can be thought of as that set of outcomes whose occurence makes an infinitely large subset of the events in the sequence happen. The Borel-Cantelli Lemmas help us characterize the probability of A_n i.o.

Theorem 1.6 (Borel-Cantelli Lemma 1) *Given a probability space* $(\Omega, \mathcal{F}, p)$ *and a sequence* $\{A_n\}$ *such that* $\forall i : A_i \in \mathcal{F}$*, if*

$$\sum_{i=1}^{\infty} p(A_i) < \infty$$

then $p(A_n \text{ i.o.}) = 0$.

Proof.

$$\begin{aligned} p(A_n \text{ i.o.}) &= p(\bigcap_{n \geq 1} \bigcup_{j \geq n} A_j) \\ &= \lim_{n \to \infty} p(\bigcup_{j \geq n} A_j) \end{aligned}$$

This is because probability is continuous, so we can say that the probability of the limit is the same as the limit of the probability.

Now using the sub-additivity property of the probability measure we can say that:

$$\begin{aligned} \lim_{n \to \infty} p(\bigcup_{j \geq n} A_j) &\leq \lim_{n \to \infty} \sum_{k=n}^{\infty} p(A_k) \\ &= 0 \end{aligned}$$

This follows from the assumption that $\sum_{i=1}^{\infty} p(A_i)$ is a finite quantity. ∎

The converse of this result is not necessarily true, but the following partial converse holds.

Theorem 1.7 (Borel-Cantelli Lemma 2) *Given a probability space* $(\Omega, \mathcal{F}, p)$ *and a sequence* $\{A_n\}$ *such that* $\forall i : A_i \in \mathcal{F}$ *and all the* A_i*s are independent of each other and*

$$\sum_{i=1}^{\infty} p(A_i) = \infty$$

then $p(A_n \ i.o.) = 1$.

Proof. We will use the notation that $A^c = \Omega \setminus A$. From the definition of lim sup and lim inf we have

$$\begin{aligned} 1 - p(\lim_{n\to\infty} \sup_{k\geq n} A_k) &= p(\lim_{n\to\infty} \sup_{k\geq n} A_k^c) \\ &= p(\bigcup_{n\geq 1} \bigcap_{k\geq n} A_k^c) \\ &= \lim_{n\to\infty} p(\bigcap_{k\geq n} A_k^c) \end{aligned}$$

This last step follows because the sequence $B_n = \bigcap_{k\geq n} A_j^c$ is a non-decreasing sequence, so it must have a limit. And, since probability is continuous, we can take the limit out.

Now, let us look at the right hand side of the last equation in a different way:

$$\begin{aligned} \lim_{n\to\infty} p(\bigcap_{k\geq n} A_k^c) &= \lim_{\ell\to\infty} p(\bigcap_{k\geq n}^{\ell} A_k^c) \\ &= \lim_{\ell\to\infty} \prod_{k=n}^{\ell} p(A_k^c) \end{aligned}$$

The second step follows from the independence assumption.

$$\begin{aligned} \lim_{\ell\to\infty} \prod_{k=n}^{\ell} p(A_k^c) &= \lim_{\ell\to\infty} \prod_{k=n}^{\ell} (1 - p(A_k)) \\ &\leq \lim_{\ell\to\infty} \prod_{k=n}^{\ell} e^{-p(A_k)} \\ &= \lim_{\ell\to\infty} e^{-\sum_{k=n}^{\ell} p(A_k)} \\ &= 0 \end{aligned}$$

We used the inequality $1 - x \leq e^{-x}$ in this sequence of deductions. And we inferred the last step because the sum of the probabilities of the events A_n

onwards must diverge, otherwise our assumption that the sum of probabilities of all the A_is diverges is contradicted. ■

Let us now consider an application of these lemmas. Borrowing from John Tsitsiklis's online course materials [3] let us extend Example 1.5.

Example 1.8 *Toss infinitely many independent coins. Each coin comes up heads with probability p. The event A_k occurs if we see a sequence of k consecutive heads in the coin tosses numbered $2^k, 2^k+1, 2^k+2, \ldots, 2^{k+1}-1$. Prove that*

$$P(A_k,\ i.o.) = \begin{cases} 0 & \text{if } p \leq 1/2, \\ 1 & \text{if } p \geq 1/2. \end{cases}$$

Let us define E_i to be the event that there are k consecutive heads beginning at coin toss number $2^k + (i-1)$. Hence $A_k = \cup_{i=1}^{2^k-k} E_i$. Hence, by subadditivity, we have that

$$\begin{aligned} \mathrm{P}(A_k) &\leq \sum_{i=1}^{2^k-k} \mathrm{P}(E_i) \\ &= \sum_{i=1}^{2^k-k} p^k \\ &\leq (2p)^k \end{aligned}$$

Hence, if $p < 1/2$,

$$\sum_{k=1}^{\infty} \mathrm{P}(A_k) \leq \sum_{k=1}^{\infty} (2p)^k < \infty$$

and, so, by the Borel-Cantelli lemma, we have proved the first part of our assertion.

For the second part we will attempt to use the partial converse of the Borel-Cantelli lemma i.e. Theorem 1.7. Note, that because of the independence of the coin tosses our A_is are independent since they take into account non-overlapping sets of coin tosses.

In order to use Theorem 1.7 we will have to show that $\sum_{k=1}^{\infty} \mathrm{P}(A_k)$ diverges for $p \geq 1/2$. Let us first consider the case $p = 1/2$. If we can prove this sum diverges for this value of p, it must diverge for all values greater since the probability of each of these events increases as the probability of a coin coming up heads increases.

So, let us try to lower bound $\sum_{k=1}^{\infty} \mathrm{P}(A_k)$. We consider the events E_i defined above and use some of these to define the event $\tilde{A}_k \subset A_k$ as follows

$$\tilde{A}_k = E_1 \cup E_{k+1} \cup E_{2k+1} \cup \cdots \cup E_{\ell k+1}.$$

where $\ell = \lfloor (2^{k+1} - k)/k \rfloor$.

Since $\mathrm{P}(A_k) \geq \mathrm{P}(\tilde{A}_k)$, it is sufficient to show that $\sum_{k=1}^{\infty} \mathrm{P}(\tilde{A}_k)$ diverges.

$$\begin{aligned}
\mathrm{P}(\tilde{A}_k) &= 1 - \mathrm{P}(\tilde{A}_k{}^c) \\
&= 1 - (1 - p^k)^{\ell} \\
&\geq 1 - (1 - p^k)^{2^k/k} \\
&= 1 - (1 - (1/2)^k)^{2^k/k} \\
&\geq 1 - (e^{2^{-k}})^{2^k/k} \\
&= 1 - e^{-1/k}
\end{aligned}$$

In the penultimate step we have used the inequality $1 - x \leq e^{-x}$. Now, we further use the Taylor expansion of e^{-x} to obtain the following bound: $e^{-1/k} \leq 1 - 1/k + (1/(2k^2)$. Using this we finally get

$$\begin{aligned}
\sum_{k=1}^{\infty} \mathrm{P}(A_k) &\geq \sum_{k=1}^{\infty} \left(1 - \left(1 - \frac{1}{k} + \frac{1}{2k^2}\right)\right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{2k^2}\right).
\end{aligned}$$

Since this last term is unbounded, the sum of the probabilities is unbounded and hence the result follows.

1.2.4 Kolmogorov's Zero-One Law

Given a set $A \subset \Omega$, we can generate a σ-field from it by adding putting A into my set, then adding $\Omega, \emptyset$ and A^c. Similarly if I had a collection of subsets $A_1, A_2, \ldots, A_n$, we can generate a σ-field from them by adding their complements and the various unions and intersections that are required to satisfy the definition of a σ-field. Let us denote the σ-field generated in this way as $\sigma(A_1, A_2, \ldots, A_n)$.

Now, let us consider a probability space $(\Omega, \mathcal{F}, p)$ and a sequence of events $\{A_n\}$ from $\mathcal{F}$. Note that the σ-field $\sigma(A_2, A_3, \ldots)$ is a subset of the σ-field $\sigma(A_1, A_2, \ldots)$, since the latter contains all the events the former contains and one more. This gives us a non-increasing sequence of σ-fields:

$$\mathcal{F}_i = \sigma(A_i, A_{i+1}, \ldots)$$

We can now define the *tail σ-field* generated by the sequence $\{A_n\}$ as

$$\mathcal{F}^* = \bigcap_{i=1}^{\infty} \mathcal{F}_i$$

We are now ready to state Kolmogorov's zero-one law.

Theorem 1.9 (Kolmogorov's Zero-One Law) *Given a probability space $(\Omega, \mathcal{F}, p)$ and a sequence $\{A_n\}$ of independent events taken from this space, if we consider an event S from the tail σ-field, $\mathcal{F}^*$ generated by the events of $\{A_n\}$, then $p(S)$ is either 0 or 1.*

Proof. Let us begin by recalling that two sigma fields $\mathcal{F}_1$ and $\mathcal{F}_2$ defined on a set Ω with a common measure p are said to be independent if for every $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, $p(A \cap B) = p(A) \cdot p(B)$ i.e. every event in $\mathcal{F}_1$ is independent of every event in $\mathcal{F}_2$.

Now note that if we have two collections of events $\mathcal{A}$ and $\mathcal{B}$ such that the events of $\mathcal{A}$ are independent of the events of $\mathcal{B}$ then the σ-fields $\sigma(\mathcal{A})$ and $\sigma(\mathcal{B})$ are independent.

Now, let us consider an event $S \in \mathcal{F}^*$. For any index n it must be true that $S \in \mathcal{F}_{n+1} = \sigma(A_{n+1}, A_{n+2}, \ldots)$. Hence S must be independent of the events $A_1, A_2, \ldots, A_n$, since these events are independent of the events in $\mathcal{F}_{n+1}$. Since this is true for any index n, we claim that $\sigma(S)$ is independent of $\sigma(A_1, A_2, \ldots)$.

However we know that S is in both $\sigma(S)$ and $\sigma(A_1, A_2, \ldots)$. Hence S must be independent of S i.e. $p(S \cap S) = p(S)^2$ which gives us the result since $p(S \cap S) = p(S)$. ■.

Let us consider the example of an application of Kolmogorov's 0-1 law that lies at the basis of the study of percolation.

Example 1.10 *Given the grid $\mathbb{Z}^2$ and a probability p of each edge remaining open, the probability of an infinte connected component remaining is either 0 or 1.*

To use Kolmogorov's zero-one law we have to show that the existence of an infinite component is in the tail σ-field of some sequence of independent events. To define this sequence let us take all the edges in $\mathbb{Z}^2$ and order them in some way. Then we define the event

$$A_i : \text{ The } i\text{th edge is open.}$$

Note that a finite number of edges being closed can disconnect only a finite number of vertices from the rest of $\mathbb{Z}^2$. Hence, the existence or non-existence of a infinite component cannot be determined by any finite subsequence of $\{A_n\}$. So it must be in the tail σ-field $\mathcal{F}^*$. Hence, its probability must be 0 or 1.

References

[1] S. R. Broadbent and J. M. Hammersley. Percolation processes I. crystals and mazes. *Proc. Cambridge Philosophical Society*, 53:629–641, 1957.

[2] H. Kesten. The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Communications in Mathematical Physics*, 74:41–59, 1980.

[3] J. Tsitsiklis. 6.975: Fundamentals of probability. Course handouts placed online at http://web.mit.edu/~cmcaram/www/6.975/, 2004.