CS105L: Discrete Structures
I semester, 2006-07

Tutorial Sheet 13: Semester review

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Note. The problems marked + may require some time.

1. Denote by $[k]$ the set $\{1, \ldots, k\}$. Let us consider a valid vertex colouring $\phi : V(G) \to [k]$ of a graph G with k colours. Now define a relation $\mathcal{R} \subseteq [k] \times [k]$ as follows: $(i, j) \in \mathcal{R}$ if there are two distinct vertices $u, v \in V(G)$ such that $(u, v) \in E(G)$ and $\phi(u) = i$ and $\phi(v) = j$

   Is $\mathcal{R}$ reflexive for any value of k? Is it symmetric for all values of k? Is it transitive for any value of k? For what values of k is it total?

2. A *rooted tree* is a pair (G, v) where G is a tree and v is a particular vertex of $V(G)$. We define the level of every vertex of G as follows. The level of v is 0. The level of all the neighbours of v is 1. The level of all vertices at distance i from the root is i. We define a relation $\mathcal{R} \subseteq V(G) \times V(G)$ as follows: (u, v) is in $\mathcal{R}$ if the path from u to v contains vertices of strictly decreasing level.

   Prove that $\mathcal{R}$ is a partial order.

3. Show that
$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} = 2 \cos \frac{\pi}{2^{n+1}}.$$
   where there are n 2s in the expression on the left.

4. (+) (Pick's Theorem) Given a simple polygon in the plane whose vertices lie on lattice points (i.e. points with both coordinates integers), show that the area of the polygon is given by $I + B/2 - 1$, where I is the number of lattice points which lie entirely within the polygon and B is the number of lattice points which lie on the boundary of the polygon.

   Prove Pick's Theorem by induction on the number of sides of the polygon. You may assume the base case, that the theorem holds for triangles.

5. Suppose G is a graph with $2n$ vertices and $n^2 + 1$ edges, $n > 2$. Show that G must contain a triangle i.e. a K^3.

6. Suppose a musical group has 11 weeks to prepare for opening night, and they intend to have at least one rehearsal each day. However, they decide not to schedule more than 12 rehearsals in any 7-day period, to keep from getting burned out. Prove that there exists a sequence of successive days during which the band has exactly 21 rehearsals.

7. (+) Prove that given any set S of exactly 7 real numbers, there are $x, y \in S$ with

$$0 < \frac{x - y}{1 + xy} \leq \frac{1}{\sqrt{3}}$$

8. (+) We call a subset U of the vertex set of a graph G *connected* if the subgraph induced by U is a connected graph. Prove that the number of connected sets in G of size r is at most $n \cdot \Delta^{2r-1}$ where n is the number of vertices in G and Δ is the maximum degree of G.

   Hint. Argue that every rooted tree can be visualized as a walk starting and ending at the root going over every edge at most twice.