CS105L: Discrete Structures
I semester, 2005-06

Tutorial Sheet 2: Sets and Mathematical Induction

Instructor: Amitabha Bagchi

August 7, 2005

Some of the problems in this sheet are from Bogart, Drysdale and Stein's book. The numbers in bold in the parentheses indicate the location of the problem in that book.

1. (**4.1, Prob 3**) Use induction to prove that
$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$
2. (**4.1, Prob 6**) Prove that
$$\sum_{i=j}^{n} \binom{i}{j} = \binom{n+1}{j+1}$$
3. (**4.1, Prob 7**) Prove that every number greater than 7 is a sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.
4. (**4.1, Prob 7**) Prove by induction that the number of subsets of an n-element set is 2^n.
5. For this problem we need some definitions.

   Definition 2.1 *A relation R on a set A (i.e. a subset of $A \times A$) is called a* partial order *if it is*

   - *reflexive:* $\forall a \in A : (a, a) \in R$.
   - *antisymmetric:* $\forall a, b \in A : ((a, b) \in R \wedge (b, a) \in R) \rightarrow a = b$.
   - *transitive:* $\forall a, b, c \in A : ((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R$.

   A partial order is denoted $\preceq$ and a partially ordered set or poset *A with a partial order on it is denoted $(A, \preceq)$.*

   Definition 2.2 *Let $(A, \preceq)$ be a poset. If every two elements of A are comparable then $\preceq$ is called a* total order *and A is called a* totally ordered set.

   Definition 2.3 *A poset $(A, \preceq)$ is called a* well-ordered set *(and $\preceq$ is called a* well-order*) if $(A, \preceq)$ is a totally ordered set and every nonempty subset of A has a least element.*

   Now the exercise. The following theorem is known as The Principle of Well-Ordered Induction. Prove it.

Theorem 2.1 *Given a well-ordered set* $(A, \preceq)$ *and a predicate* $p(x)$*, suppose that* A *is non-empty and* x_0 *is the least element of* A*. Show that if*

- *Basis:* $p(x_0)$ *is true and*
- *Induction step: For every* $y \in A$*, if* $p(x)$ *holds for all* $x \preceq y$ *then* $p(y)$ *holds.*

Then $p(x)$ *holds for all* $x \in A$*.*