## 7.6. Partial orders

One often wants to define or use an ordering relation on a set. Or compare elements under some notion of size or priority. Or prove something by induction on a suitable ordering of a set.

Such orders are relations: the relation asserts that one element of the set is
bigger
more complicated
of lower degree
of higher prioity
first in some (e.g. lexicographic) order (A "precedes" B)
than some other element.

A key property of such a relation is that it be *antisymmetric*. Symmetry is exactly what you do not want.

**Definition.** A relation $R$ on a set $A$ is called a *partial order* if it is:
reflexive
antisymmetric
transitive.

A partial order is also called a partial ordering. A pair $(A, R)$ consisting of a set $A$ together with a partial order $R$ on $A$ is called a *partially ordered set* or *poset*.

The symbol $\preceq$ is often used to denote a partial order $R$, where $a \preceq b$ means that $(a, b) \in R$.

This is because the relation $\leq$ on $\mathbb{R}$ is a paradigm for the partial order notion. But we will use $\preceq$ to denote a general partial order on a set.

**Examples**

1. $(\mathbb{R}, \leq)$. Not: $(\mathbb{R}, <)$
2. $(\mathbb{R}, \geq)$
3. $(\mathbb{Z}, |)$ (if we stipulate $0|0$), or $(\mathbb{Z}^+, |)$.
4. $(\mathbb{Z}, \leq)$
5. Not: $(\mathbb{Z}, \equiv \bmod m)$
6. Lexicographic order on $\mathbb{Z}^2$
7. Lex order on strings of letters

We have defined a partial order to be reflexive, so always $a \preceq a$. However, since we assume it is antisymmetric we know that if $a \preceq b$ and $b \preceq a$ then in fact $a = b$. So we can recover the "strict" (nonreflexive) order (like $<$) by setting $a \prec b$ to mean that $a \preceq b$ and $a \neq b$.

Alternately, if we took our basic notion to be an *irreflexive* (antisymmetric transitive) order $\prec$ we could recover $\preceq$ as $\prec$ or $=$.

While the poset $(\mathbb{R}, \leq)$ is a paradigm, it is very special. In particular it has the property that for any two elements $a, b \in \mathbb{R}$, either $a \leq b$ or $b \leq a$.

(For $<$ there are three possibilities: either $a < b, b < a, a = b$, one reason to use the reflexive order $\leq$.)

We do not require this of a general partial order. And, e.g. in the poset $(\mathbb{Z}, a|b)$ there are pairs such as $5, 6$ such that neither $5|6$ nor $6|5$.

That is in fact the import of the word "partial".

So we make the following definitions.

**Definition.** Let $(A, \preceq)$ be a poset, and $a, b \in A$. We say that $a, b$ are *comparable* if either $a \preceq b$ or $b \preceq a$ (i.e. if the order orders them!). Otherwise we call $a, b$ *incomparable*.

**Example.** In the poset $(\mathbb{Z}, |)$:
$3, 6$ comparable
$6, 3$ comparable
$5, 7$ incomparable

**Definition.** Let $(A, \preceq)$ be a poset. If every two elements of $A$ are comparable, then $\preceq$ is called a *total order* or *total ordering* and the set $A$ is called a *totally ordered set* or a linearly ordered set. (I will stick to the word "total").

**Example**
1. $\leq$ is a total order on $\mathbb{R}$
2. $|$ is a not a total order on $\mathbb{Z}$ (or $\mathbb{Z}^+$)
3. Lex is a total order on $\mathbb{Z}^2$ (or $\mathbb{Z}^3$ etc)
4. Lex is a total order on strings of letters.

This brings us to another important notion.

**Definition.** A poset $(A, \preceq)$ is called a *well-ordered set* (and $\preceq$ is called a *well-order*) if $(A, \preceq)$ is a totally ordered set and every nonempty subset of $A$ has a least element.

Here "least element" of a subset $B \subset A$ means: an element $b \in B$ such that $b \preceq c$ for every $c \in B$.

Note that a "least element", if it exists, is unique: for if two elements $b, d \in B$ have the above property then:

$b \preceq d$ becasue $b$ is "least" and $d \in B$,

but likewise

$d \preceq b$ because $d$ is "least" and $b \in B$
so $b = d$ by the antisymmetry of $\preceq$

**Examples**

1. Our erstwhile paradigm $(\mathbb{R}, \leq)$ is **not** a wellorder.

E.g. the set $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ has no least element. (That is why one defines limsup and liminf).

In fact Cantor, the inventor of set theory and cardinality, spent a lot of time trying to construct a wellorder of $\mathbb{R}$.

And thereby hangs a tale.

2. Lex (on $\mathbb{Z}^2$, $\mathbb{Z}^3$, etc, or on letter strings) is a well order. We proved this for $\mathbb{Z}^2$ in our study of generalized induction.

**Theorem.** (The principle of wellordered induction).

*Suppose that $(A, \preceq)$ is a well ordered set and $P(x)$ is a predicate. Suppose $A$ is nonempty (!) and that $x_0$ is the least element of $A$.*

*Suppose*

*1. Basis step: $P(x_0)$ is true and*

*2. Induction step: For every $y \in A$, if $P(x)$ holds for all $x \prec y$ then $P(y)$ holds.*

*Then $P(x)$ holds for all $x \in A$.*

**Proof.**

We prove it by contradiction!

Suppose that $P(x)$ does **not** hold for all $x \in A$.

Then the subset $B = \{x \in A : \neg P(x)\}$ of $A$ is nonempty.

Therefore (because $\preceq$ is a wellorder) $B$ has a least element $a$.

So $\neg P(a)$ holds, but $P(b)$ holds for all $b \prec a$.

So $a \neq x_0$ (since $P(x_0)$ holds). And the inuction step now implies that $P(a)$ holds. Contradiction.

So indeed $P(x)$ holds for all $x \in A$. □

Remarks

1. We already did this proof in specific cases. Here we see it in its natural setting of a wellordered set.

2. This general induction principle is very versatile.

3. Hence the importance of wellorders.

**Lexicoghraphic order**

The words in a dictionary are listed in *alphabetic* or *lexicographic* order. This order is an ordering of strings on the alphabet that is inherited from the ordering of the alphabet ("a,b,c,...,z").

A familiar but nontrivial and important process that we will now study.

First we look at how to put a partial order on the cartesian product of two posets $(A_1, \preceq_1), (A_2, \preceq_2)$. So these are just two sets, each having a partial order.

**Definition.** Suppose $(A_1, \preceq_1), (A_2, \preceq_2)$ are posets. the *lexicographic order* $\preceq$ on $A_1 \times A_2$ is given by specifying that

$(a_1, a_2) \prec (b_1, b_2)$ if
$a_1 \prec_1 b_1$ or $a_1 = b_1$ and $a_2 \prec b_2$

and then we put

$(a_1, a_2) \preceq (b_1, b_2)$ if
$(a_1, a_2) \prec (b_1, b_2)$ or $(a_1, a_2) = (b_1, b_2)$

Note. We had to be a bit careful. If you just say: $(a_1, a_2) \preceq (b_1, b_2)$ if $a_1 \preceq_1 b_1$ or ... you do not get what you want because then $(a, b) \preceq (a, c)$ for any $b, c$, which we do not want.

**Example:** In the poset $\mathbb{Z} \times \mathbb{Z}, \preceq$ constructed from the poset $(\mathbb{Z}, \leq)$:
$(3, 4) \preceq (5, 1)$
$(3, 4) \preceq (5, -3456)$ indeed
$(3, 4) \prec (5, -3456)$
$(3, 4) \not\preceq (2, 1)$
$(2, 1) \preceq (3, 4)$ (it is a wellorder)
$(3, 4) \preceq (3, 7)$, indeed
$(3, 4) \prec (3, 7)$

One similary defined the lexicographic order on a product

$$A_1 \times A_2 \times \ldots \times A_n$$

where $(A_1, \preceq_1), (A_2, \preceq_2), \ldots, (A_n, \preceq_n)$ are posets:

$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$ if

$a_1 \prec_1 b_1$ or
$a_1 = b_1$ and $a_2 \prec_2 b_2$
or
$a_1 = b_1, a_2 = b_2, \ldots, a_i = b_i$ and
$a_{i+1} \prec_{i+1} b_{i+1}$

and then
$(a_1, a_2, \ldots, a_n) \preceq (b_1, b_2, \ldots, b_n)$ if
$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$ or
$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

**Example**

1. In the lex order on $\mathbb{Z}^5$:

$$(1, 2, 1, 56, 345) \prec (1, 2, 3, 4, 5)$$

$$\prec (1,2,3,4,6) \prec (1,3,-500,-234,-167).$$

2. There is likewise a lex order on $\mathbb{R}^2$ or $\mathbb{R}^5$, but it is not a wellorder.

**Lex order on strings**

This is like lex order on cartesian products, the new possibility is that we need to order strings of different length.

And simply say: the shorter string is first if all the characters of the short string agree with those of the long string.

Thus in the poset (strings, $\preceq$) built from the order on the alphabet $a, b, c, \ldots, z$:

$$memorandum \prec memorize \prec memory$$

$$to \prec top \prec topology$$

$$a \prec at \prec ate.$$

**Hasse diagrams**

We saw that one can draw a directed graph corresponding to a partial order.

E.g. for $(\{1,2,3,4\}, \leq)$

Or for $(\{1,2,3,4,6,8,12\}, |)$.

If one knows (or assumes) the order is a partial order (so reflexive, transitive), one does not need to draw the edges which have to be there by those properties and other edges present. Thus:

One can remove loops, any edge that is there by transitivity. We can also do the picture so that edges all go up hill (at least a bit), and no need to show the arrows.

**Examples** (draw these!)

1. $(\{1,2,3,4\}, \leq)$
2. $(\{1,2,3,4,6,8,12\}, |)$.
3. $(P(\{1,2,3\}), \subseteq)$.
4. $(\mathbb{Z}, \leq)$

**Maximal and minimal elements**

**Definition.** Let $(A, \preceq)$ be a poset. An element $a \in A$ is called *maximal* if there is no element $b \in A$ such that $a \prec b$.

An element $a \in A$ is called *minimal* if there is no element $b \in A$ such that $b \prec a$.

These are easy to spot on a Hasse diagram: they are elements at "top" (or with nothin above), or "bottom" (nothing below).

And they don't always exists: e.g. the poset $(\mathbb{Z}, \leq)$ has neither.

And there may be more than one. In the poset $(\{1, 2, 3, 4, 6, 8, 12\}, |)$, the elemnt 12 is a maximal element. So is 8.

**Definition.** Let $(A, \preceq)$ be a poset. An element $a \in A$ is called *greatest* if $b \preceq a$ for every $b \in A$.

An element $a \in A$ is called *least* if $a \preceq b$ for every $b \in A$.

**Examples**

1. In $(\{1, 2, 3, 4, 6, 8, 12\}, |)$ there is no greatest element, but 1 is the least element.

2. In $P(\{1, 2, 3\}, \subseteq)$, $\{1, 2, 3\}$ is greatest, $\emptyset$ is least.

So greatest and least elements need not exists, but if they exist they are unique.

**Theorem.** *Let $(A, \preceq)$ be a poset. Suppose $a, b$ are greatest element of $A$. Then $a = b$.*

**Proof.**
Since $a$ is greatest, $b \preceq a$.
Since $b$ is greatest, $a \preceq b$.
Therefore, by the antisymmetry property of a partial order,
$a = b$. □

**Theorem.** *Let $(A, \preceq)$ be a poset. Suppose $a, b$ are least element of $A$. Then $a = b$.*

The proof is very similar.

**Upper and lower bounds**

**Definition.** Let $(A, \preceq)$ be a poset, and $B \subseteq A$.

An element $u \in A$ is called an *upper bound* for $B$ if $b \leq u$ for all $b \in B$.

An element $l \in A$ is called a *lower bound* for $B$ if $l \preceq b$ for all $b \in B$.

Note that an upper bound (or lower bound) need not belong to $B$.

**Examples**

1. In $(\{1, 2, 3, 4, 6, 8, 12\}, |)$,
12 is an upper bound for $\{1, 2, 4\}$
so is 8
so is 4.
but not 6 or 3.
6 is an upper bound for $\{1, 2, 3\}$
6 is an upper bound for $\{1, 2, 3, 6\}$.
4 is a lower bound for $\{4, 8, 12\}$

so is 2
so is 1
but not 3 or 6.

**Definition.** Let $(A, \preceq)$ be a poset, and $B \subseteq A$.

An element $u \in A$ is called a *least upper bound* of $B$ if it is an upper bound for $B$, and is least among upper bounds, i.e. if $v$ is an upper bound for $B$ then $u \preceq v$.

An element $l \in A$ is called a *greatest lower bound* of $B$ if iot s a lower bound and greatest among lower bounds, i.e. if $m$ is a lower bound for $B$ then $m \preceq l$.

Again, the greatest lower bound and least upper bound for $B$ need not belong to $B$.

And need not exist: maybe there are no upper bounds to begin with.

**Theorem.** *Let $(A, \preceq)$ be a poset, and $B \subseteq A$.*

*The greatest lower bound of $B$ , if it exists, is unique.*

*The least upper bound of $B$, if it exists, is unique.*

**Proof.**

**Examples**

1. In $(\mathbb{Z}, |)$.

What is the greatest lower bound of a subset $B$?
$l$ a lower bound if it divides every element of $B$.
I.e.: a lower bound is a common divisor.
A greatest common divisor means: it divides every element of $B$, but every other common divisisor divides it.
I.e. it is a gcd.
We have proved this always exists.

2. In $P(X), \subseteq)$

A lower bound for (say) two subsets $A, B \subset X$ is a set $C$ such that $C \subseteq A$ and $C \subseteq B$.

The empty set $\emptyset$ is (always) a lower bound for $\{A, B\}$..
But the greatest lower bound is (prove this!)
$A \cap B$.
Similarly for more sets
$A, B, C, D, \ldots \subseteq X$.
Least upper bound: the union.

**Lattices**

**Definition.** A poset in which every pair of elements has a greatest lower bound and a least upper bound is called a lattice.

**Examples**

1. $(\mathbb{Z}^+, |)$ is a lattice.

2. $(P(X), \subseteq)$ is a lattice.

3. $(\{1, 2, 3, 4, 5\}, |)$ is not a lattice
(The pair 4,5 has no upper bound.)

4. $(\{1, 2, 4, 8, 16\}, |)$ is a lattice