COL202: Discrete Mathematical Structures. I semester, 2022-23.
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Tutorial Sheet 8: Infinite sets.
20 October 2022

Important: The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a * are optional challenge problems and are not to be discussed in the tutorial.

Problem 1
In class we discussed Cantor's predicament when a fleet of buses indexed by $\mathbb{N}$ appeared at his hotel, each with seats indexed by $\mathbb{N}$. We suggested that Cantor can accommodate the guests as follows

- Currently resident guest in room i can shift to room 2^i.
- The j passenger of bus k can move into room p_{k+1}^j where p_k denotes the kth prime number, i.e., $p_1 = 2$.

Problem 1.1
Extend the solution discussed in class to the case where the fleet of buses is indexed by $\mathbb{N}^2$.

Problem 1.2
Now let us consider a situation where the government declares that only rooms whose number is a multiple of 6 can be used. What should Cantor do to accomodate the guests now? In this case assume the buses are indexed by $\mathbb{N}$.

Problem 1.3
Use the solution of Problem 1.2 to give a simple way of accomodating a fleet of buses indexed by $\mathbb{N}^k$ for any $k \geq 1$.

Problem 2
Which of the following is true?

1. A strict $\mathbb{N} \Leftrightarrow A$ is finite.
2. A strict $\mathbb{N} \Leftrightarrow \mathbb{N}$ $\mathsf{surj} A$.
3. A strict $\mathbb{N} \Leftrightarrow \exists n \in \mathbb{N} : |A| < n$.

Problem 3 ♠ [1]
Prove that the set $\{0, 1\}^*$ of all finite strings on 0 and 1 is countable.

Problem 4
Prove that the set of positive rationals is countable. Then extend this proof to show that the set of rationals is countable.

Problem 5 [1]
Prove that the set $\mathbb{N}^*$ of all finite sequences of natural numbers is countable.

Problem 6 [1]
Suppose we have an infinite sequence $\{f_i\}_{i \geq 1}$ of functions from $\mathbb{N}$ to $\mathbb{R}_+$ (positive reals). A function $h : \mathbb{N} \to \mathbb{R}_+$ is said to *majorize* the sequence if for each $k \in \mathbb{N}$ there is some $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 : f_k(n) \leq h(n)$.

Problem 6.1
Before going to the main question do the following

1. Show that for any finite $A \subset \mathbb{R}_+$, $\sup A \in \mathbb{R}_+$.
2. Show that there exist infinite sets of the form $A \subset \mathbb{R}_+$ such that $\sup A \notin \mathbb{R}_+$ (i.e. the supremum is ∞).

Problem 6.2
Give an explicit construction for h using Problem 6.1 as a hint. Is there some way of doing it without using this hint?

Problem 6.3
Also show that there is an h such that $f_k(n)$ is $o(h(n))$ for every $k \in \mathbb{N}$.

Problem 7
In class we proved the Schroder-Bernstein theorem using Tarski's Fixed Point Theorem. However the better known proof proceeds more explicitly. First try to work out a proof for the finite case. Then go and look at the structure of the proof for the general case provided in Problem 8.14 of [1]. Work out the proof according to the directions provided there. (It's very long so I am not copying it all out here).

Problem 8 *
Suppose we are given a graph $G = (V, E)$ where V is an infinite set. We say that such a graph is connected if there is a finite length path between any two vertices. Prove that every connected graph has a spanning tree. (Hint: Consider the poset of the trees contained in G ordered by the subgraph relation and see if you can apply Zorn's Lemma to prove the result.)

Problem 9
Prove that $\mathbb{R}$ is uncountable.

Problem 10 [1]
An infinite binary string is called OK if the 1s are only allowed to appear in perfect square positions, i.e., at positions $1, 4, 9, \ldots$. Note that not all the perfect square positions must be 1, but all non-perfect square positions must be 0. Prove that the set of OK strings is uncountable.

References

[1] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science June 2018, MIT Open Courseware.