COL202: Discrete Mathematical Structures. I semester, 2022-23.
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Tutorial Sheet 10: Generating functions.
3 November 2022

Important: The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a * are optional challenge problems and are not to be discussed in the tutorial.

Problem 1
In the 2-dimensional plane we have n lines such that no two lines are parallel and no three lines intersect at one point. If R_n is the number of regions created by these n lines, find a recurrence for R_n and solve it.

Problem 2
Find a recurrence relation for the number of bit strings of length n that contain the string 01. Try and solve it if possible.

Problem 3
Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s. Try and solve it if possible.

Problem 4
Let A_n be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of A_n . Solve this recurrence relation to find a formula for d_n.

Problem 5
In how many ways can $3r$ balls be chosen from $2r$ red balls, $2r$ blue ballls and $2r$ green balls?

Problem 6
Evaluate the following sums:

Problem 6.1

$$\binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + i \cdot \binom{n}{i} + \cdots + n \cdot \binom{n}{n}$$

Problem 6.2
Given that $k \leq m$ and $k \leq n$

$$\binom{n}{0} \cdot \binom{m}{k} + \binom{n}{1} \cdot \binom{m}{k-1} + \binom{n}{2} \cdot \binom{m}{k-2} + \cdots + \binom{n}{k} \cdot \binom{m}{0},$$

Problem 6.3

$$\binom{2n}{n} + \binom{2n-1}{n-1} + \cdots + \binom{2n-i}{n-i} + \cdots + \binom{n}{0}$$

Problem 7 ♠ (pp 25 of [1])
Let X be a random variable that takes values $0, 1, 2, \ldots$ with probabilities $p_0, p_1, p_2, \ldots$ respectively. Clearly we must have p_i is nonegative for each i and the sum of p_is is 1. Let $P(x)$ be the ordinary power series generating function (opsgf) of $\{p_n\}_{n \geq 0}$.

Problem 7.1
Express the mean and standard deviation of X in terms of $P(x)$.

Problem 7.2
Let X_1 and X_2 be two independent random variables with the same distribution as X. Let $p_n^{(2)}$ be the probability that $X_1 + X_2 = n$. What is the opsgf of $\{p_n^{(2)}\}_{n\geq 0}$?

Problem 7.3
For $k \geq 2$, let $X_1, \ldots, X_k$ be k independent random variables with the same distribution as X. Let $p_n^{(k)}$ be the probability that $\sum_{i=1}^{k} X_i = n$. What is the opsgf of $\{p_n^{(k)}\}_{n\geq 0}$?

Problem 7.4
Use the results above to write out the mean and standard deviation of $\sum_{i=1}^{k} X_i$ where the X_i are independently chosen with the same distribution as X.

Problem 8
Let $f(n, k, h)$ be the number of ordered representations of n as a sum of exactly k integers each of which is $\geq h$. Find the generating function $\sum_n f(n, k, h)x^n$. By ordered representation we mean that that if $n = 10$, $k = 3$ and $h = 2$ then we will consider $5 + 3 + 2$ and $2 + 3 + 5$ as two *different* representations.

Problem 9
In each part below the sequence $\{a_n\}_{n\geq 0}$ satisfies the given recurrence. Find the ordinary power series generating function in each case and solve to find a_n where possible.

Problem 9.1

$$a_{n+1} = 3a_n + 2, (n \geq 0, a_0 = 0).$$

Problem 9.2

$$a_{n+1} = \alpha a_n + \beta, (n \geq 0, a_0 = 0).$$

Problem 9.3

$$a_{n+2} = 2a_{n+1} - a_n, (n \geq 0, a_0 = 0, a_1 = 1).$$

Problem 9.4

$$a_{n+1} = a_n/3 + 1, (n \geq 0, a_0 = 0).$$

Problem 10
Let $f(n)$ be the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive integers. Find a recurrence for $f(n)$ and try to solve it to the extent possible using generating functions.

Problem 11
In the following assume that $A(x)$, $B(x)$ and $C(x)$ are the ordinary power series generating functions of the sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ respectively. With this notation attempt the following problems:

Problem 11.1
If $c_n = \sum_{j+2k\leq n} a_j b_k$, express $C(x)$ in terms of $A(x)$ and $B(x)$.

Problem 11.2
If

$$nb_n = \sum_{k=0}^{n} 2^k \frac{a_k}{(n-k)!},$$

express $A(x)$ in terms of $B(x)$.

Problem 12
Solve the recurrence $g_0 = 0, g_1 = 1$ and

$$g_n = -2ng_{n-1} + \sum_{k=0}^{n} \binom{n}{k} g_k g_{n-k}, \text{ for } n > 1,$$

using an exponential generating function.

References

[1] Herbert S. Wilf, generatingfunctionology, 1994, Academic Press.