Homework III

Due on April 7, 2024

1. **([TB] Q 33.2)** Suppose we run the Arnoldi iteration on an $m \times m$ matrix A and it terminates with $h_{n+1,n} = 0$ for some $n < m$ (i.e., the vector $q_{n+1} = 0$). Let K_n denote the Krylov space spanned by $\langle b, Ab, \ldots, A^{n-1}b \rangle$.

   (i) **(3 marks)** Show that $K_n = K_{n+1} = K_{n+2} = \ldots$.

   (ii) **(3 marks)** Show that each eigenvalue of the matrix H_n is an eigenvalue of A.

   (iii) **(4 marks)** Show that if A is invertible, then the solution x to $Ax = b$ lies in K_n.

2. Let A and B be real $n \times n$ symmetric matrices such that $AB = BA$.

   (i) **(5 marks)** Suppose A has n distinct eigenvalues. Show that there are diagonal matrices D_1 and D_2 and an orthogonal matrix Q such that $A = QD_1Q^T$ and $B = QD_2Q^T$.

   (ii) **(5 marks)** Now prove the same statement as above but without the assumption that A has n distinct eigenvalues.

3. Let A be a real $n \times n$ symmetric matrix. Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A.

   (i) **(5 marks)** For a subspace S, let $M(S)$ denote $\max_{u \in S, |u|=1} u^T A u$, and let $N(S)$ denote $\min_{u \in S, |u|=1} u^T A u$. Show that

   $$\lambda_k = \max_S N(S) = \min_T M(T),$$

   where the maximum is taken over all subspaces of dimension k and the minimum is taken over all subspaces of dimension $n - k + 1$.

   (ii) **(10 marks)** Suppose A is of the form $\begin{pmatrix} H & b^T \\ b & u \end{pmatrix}$, where H is an $(m-1) \times (m-1)$ symmetric matrix. Let the eigenvalues of H be $\mu_1 \geq \ldots \geq \mu_{n-1}$. Show that for each $i \leq n-1$, $\mu_i \leq \lambda_i$, and $\mu_i \geq \lambda_{i+1}$.

4. Let A be an $m \times m$ complex matrix.

   (i) **(4 marks)** Suppose $A^*A = AA^*$. Show that there is a unitary matrix X such that X^*AX is a diagonal matrix.

   (ii) **(3 marks)** Let $W(A)$ denote the set of all values $\frac{u^*Au}{u^Tu}$, where u is a non-zero (complex) vector. Show that $W(A)$ contains the convex hull (in the complex plane) of all the eigenvalues of A.

   (iii) **(3 marks)** Suppose the matix A satisfies the condition in (i) above. Then show that $W(A)$ is equal to the convex hull of the eigenvalues of A.

5. **(Heath 5.18)**

   (a) **(5 marks)** Write a method based on Newton's method to solve the following system of non-linear equations:

   $$\begin{aligned}(x_1+3)(x_2^3-7)+18 &= 0\\ \sin(x_2 e^{x_1}-1) &= 0\end{aligned}$$

   with the starting point $x_0 = [-0.5\ 1.4]^T$.

   (b) **(5 marks)** Solve the above problem using Broyden's method.

   (c) **(3 marks)** Compare the convergence rates of the two methods by computing error at each iteration, given that the exact solution is $[0\ 1]^T$. How many iterations does each method require to achieve error close to the machine precision?

6. **(Heath 4.14)**The matrix exponential of an $n \times n$ matrix A is defined as

   $$e^A := I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots.$$

   (a) **(3 marks)** Write a program to compute e^A using the definition above.

   (b) **(3 marks)** Write a program based on the eigenvalue-eigenvector decomposition of A (assume it has distinct eigenvalues and you are given the eigenvalues and the eigenvectors).

   (4 marks) Test the above two methods on

   $$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} -49 & 24 \\ -64 & 31 \end{pmatrix}.$$

   Compare your results with the matlab function for matrix exponential. Which of the two methods is more accurate (on the above examples), and why?

7. **(10 marks)** Write a program implementing the Lanczos method. Test your method on a random symmetric matrix of order n having eigenvalues $1, 2, \dots, n$. To generate such a matrix, first generate a random $n \times n$ matrix B with random entries uniformly distributed in $[0, 1)$. Let the $B = QR$ be the QR factorization of B. Now take $A = QDQ^T$, where D is a diagonal matrix with diagonal entries $1, 2, \dots, n$. The Lanczos algorithm needs to find eigenvalues of a tridiagonal matrix – use matlab library routine to find these eigenvalues. For the purpose of this exercise, run the Lanczos iteration for a full n iterations.

   To see graphically how the Ritz values behave, construct a plot with the iteration number on the vertical axis and the Ritz value at each iteration on the horizontal axis. Plot each pair (γ, k) where γ is a Ritz value in iteration k as a discrete point. Try several values of $n = 30, 40, 50$.