Assignment 1

1. Solve the following linear programs by the simplex method (by hand):

   (a)

$$\begin{array}{rcl} \text{maximize} & & 2x_1 + x_2 \\ 2x_1 + 3x_2 & \leq & 3 \\ x_1 + 5x_2 & \leq & 1 \\ 2x_1 + x_2 & \leq & 4 \\ 4x_1 + x_2 & \leq & 5 \\ x_1, x_2 & \geq & 0 \end{array}$$

   (b)

$$\begin{array}{rcl} \text{maximize} & & 3x_1 + x_2 \\ x_1 - x_2 & \leq & -1 \\ -x_1 - x_2 & \leq & -3 \\ 2x_1 - x_2 & \leq & 2 \\ x_1, x_2 & \geq & 0 \end{array}$$

2. Use the simplex method to describe **all** the optimal solutions to the following:

$$\begin{array}{rcl} \text{maximize} & & 2x_1 + 3x_2 + 5x_3 + 4x_4 \\ x_1 + 2x_2 + 3x_3 + x_4 & \leq & 5 \\ x_1 + x_2 + 2x_3 + 3x_4 & \leq & 3 \\ x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

3. Prove or disprove: A feasible dictionary whose row about the objective function reads $z = v + \sum_{j \in N} \bar{c}_j x_j$ describes an optimal solution if and only if $\bar{c}_j \leq 0$ for all $j \in N$ (recall that N denotes the set of non-basic variables).

4. Suppose that a linear programming problem has the following property: its initial dictionary is not degenerate and, when solved by the simplex method, there is never a tie for the choice of leaving variable.

   (a) Can such a problem have degenerate dictionaries? Explain.

   (b) Can such a problem cycle? Explain.

5. Consider the following linear program where the variables are $x_1, x_2, \ldots, x_n$. Note that $b_1, b_2, \ldots, b_n$ are constants such that $1 = b_1 << b_2 << b_3 \ldots << b_n$. For example, you can take b_i to be 100^{i-1}.

$$\begin{array}{rcll} \text{maximize} & & \sum_{j=1}^{n} 10^{n-j} x_j - \frac{1}{2} \sum_{j=1}^{n} 10^{n-j} b_j & \\ 2 \sum_{j=1}^{i-1} 10^{i-j} x_j + x_i & \leq & 2 \sum_{j=1}^{i-1} 10^{i-j} b_j + b_i & \text{for } i = 1, 2, \ldots, n \\ x_j & \geq & 0 & \text{for } j = 1, 2, \ldots, n \end{array}$$

- Solve the above LP by hand for $n = 2$. Use the largest coefficient rule to decide which variable enters the set of basic variables.
- Let $w_1, w_2, \ldots, w_n$ be the slack variables (where w_i corresponds to the constraint indexed by i above) – note that we have been using $x_{n+1}, \ldots, x_{n+m}$ to denote the slack variables, but here we will use this alternate notation. Show that in any dictionary, exactly one of $\{w_i, x_i\}$ will be basic variable for all $i = 1, 2, \ldots, n$.
- Consider the dictionary

$$\begin{aligned} z &= -\sum_{j=1}^{n} \epsilon_j 10^{n-j} \left(\frac{1}{2} b_j - x_j \right) \\ w_i &= \sum_{j=1}^{i-1} \epsilon_j \epsilon_i 10^{i-j} (b_j - 2x_j) + (b_i - x_i) \quad \text{for } i = 1, 2, \ldots, n, \end{aligned}$$

  where each ϵ_i is either 1 or -1. Fix k and consider the pivot in which x_k enters the basis and w_k leaves the basis. Show that the resulting dictionary is of the same form as before. How are the new ϵ_is related to the old ϵ_is?
- Use the above result to show that the simplex method requires 2^{n-1} pivot steps in the above linear program if we follow the largest coefficient rule.