1 Answer 1

1. If there are edge-disjoints paths to S, then there are also edge-disjoint paths to any subset of S.

2. Suppose $|T| > |S|$ and there are edge disjoint paths from s to all vertices in T, and similarly for S. Create a flow instance as follows: add a new vertex t, and have edges between t and every vertex in $S \cup T$. Give capacity 1 to all edges. Since there are $|T|$ disjoint paths from s to t, max-flow from s to t has value at least $|T|$. Now let $\mathcal{P}$ be the set of edge disjoint paths from s to vertices in S – sending 1 unit of flow on these paths (and on the edges connecting S to t), we get a flow from s to t of value $|S|$. Let f be this flow, and G_f be the residual graph. Since $|T| > |S|$, we know that we can send at least one more unit of flow in G_f from s to t (recall the augmenting path algorithm – as long as we are not at a max-flow, we can always find an augmenting path in the current residual graph). So, we find a path in G_f from s to t and send 1 more unit of flow on it. This path does not contain any of the edges $(x, t), x \in S$ (because these edges are saturated), and so, in the new flow, we still saturate all the edges connecting S to t. Thus, we get a flow from s to t of value $|S| + 1$. If we now think of this flow as a set of edge-disjoint paths from s to t, we see that it has edges disjoint paths from s to every vertex in S and one more vertex in $T \setminus S$.

2 Answer 2

Given a set of vectors S, let $dim(S)$ denote the dimension of the subspace spanned by them.

2.1 Proof that $(X, \mathcal{I})$ is a matroid

1. Let $S \in \mathcal{I}$, $T \subset S$. Then, since $X \setminus S \subset X \setminus T$, $dim(X \setminus T) \geq dim(X \setminus S) = dim(X)$ (as dimension of a subset can only be less). Hence, $T \in \mathcal{I}$.

2. Let $S, T \in \mathcal{I}$, $|T| > |S|$. We need to show that we can add an element of $T \setminus S$ to S and keep it in $\mathcal{I}$. Let B_1 be a basis of $X \setminus (S \cup T)$. Now, extend B_1 to a basis B of $X \setminus S$. We first show that B cannot contain all elements of $T \setminus S$ – once we show this, we are done because take any element $t \in T \setminus S$, and add it to S. Since B does not have a common element with $S + t$, $S + t \in \mathcal{I}$ as well.

   Suppose for the sake of contradiction, $T \setminus S$ is a subset of B. Then, $|B| \geq |B_1| + |T \setminus S| > |B_1| + |S \setminus T|$ (the last inequality follows because $|T| > |S|$ and so, $|T \setminus S| > |S \setminus T|$. But we can extend B_1 to a basis of $X \setminus T$ and the size of this basis will be $|B|$. What elements can we add to B_1 during this process ? These will have to be elements in $S \setminus T$. But then, $|B| \leq |B_1| + |S \setminus T|$, which is a contradiction.

2.2 Algorithm for two disjoint bases

Let $\mathbb{L}(X)$ be the linear matroid on X, i.e., the matroid whose independent sets are linearly independent sets of vectors in X, and $\mathcal{I}$ be the independent sets of the matroid mentioned above. Then, compute a maximum size subset S in $\mathbb{L}(X) \cap \mathcal{I}$ (using martroid intersection). If $|S| = dim(X)$, then output S and a basis of $X \setminus S$ – both will have size $dim(X)$. Otherwise output that no two disjoint bases exist.

3 Answer 3

We work with two matroids – for the first matroid, we consider the matrix $A[I, C]$ – the submatrix of A obtained by considering rows in I only. The matroid $\mathcal{M}_1$ is the linear matroid on the columns of this submatrix, i.e., the set of elements are the columns of this matrix, and the independent sets are the sets of columns which are linearly independent. For the second matroid, we consider the matrix $A[R \setminus I, C]$. The matroid $\mathcal{M}_2$ has as elements the columns of the matrix $A[R \setminus I, C]$, and a subset S of columns in this matrix is independent in this matroid iff its deleting them does not change the rank of this matrix – by the previous question, this is a matroid. We will show that there is an independent set of size at least $|I|$ in both the matroid. This is enough for our purposes because if J is the set of columns corresponding to this independent set, then we are saying that J is independent in the first matroid, i.e., the square matrix $A[I, J]$ is full rank. Similarly, $A[R \setminus I, C \setminus J]$ has full rank.

To prove this statement, we use the matroid intersection theorem. It is enough to show that for every subset U of columns, $r_1(U) + r_2(C \setminus U) \geq |I|$.

Theorem 3.1 $rank(A[I, U]) + rank(A[R \setminus I, U]) \geq |U|$.

Proof Let S be the subspace spanned by vectors obtained by extending the columns of $A[I, U]$ to length $|R|$ by keeping the coordinates at indices corresponding to I same and putting zeros at the other entries. Let T be the subspace spanned by vectors obtained by extending the columns of $A(R-I, U)$ in a similar way. Then, $dim(S) = rank(A[I, U])$ and $dim(T) = rank(A[R \setminus I, U])$. Observe that $S \cup T$ spans U. Hence, $dim(U) \leq dim(S \cup T) = dim(S) + dim(T) - dim(S \cap T) = rank(A[I, U]) + rank(A[R-I, U]) - 0$, which finishes the proof. ($dim(S \cap T) = 0$ because vectors of S are zero at coordinates indexed by $R - I$ and vectors of T are zero at coordinates indexed by I, hence the only common vector is zero vector) **QED**

Let B_1 be a basis of the columns of $A[R \setminus I, U]$. Then, $|B_1| \geq |U| - r_1(U)$ by the above theorem. Extend B_1 to a basis B of $A[R \setminus I, C]$, where $|B| = |C| - |I|$. Let B_2 denote $B \setminus B_1$. Note that B_2 is a subset of $C \setminus U$. Let S be the columns of $C \setminus U$ which are not in B_2. Note that $rank(A[R \setminus I, C \setminus S]) = |R| - |I|$, because the set of columns in this sub matrix includes B. Therefore, $r_2(C \setminus U) \geq |S| \geq |I| - r_1(U)$.