Assignment 1

1. Let G be a directed graph and s be a vertex in G. Let V denote the set of vertices in G. Define a set system $(V, \mathcal{I})$ as follows (recall that $\mathcal{I}$ contains subsets of V): a set $S \in \mathcal{I}$ if there are edge-disjoint paths from s to vertices in S (i.e., there are $|S|$ paths, one from s to each vertex in S, and they are edge-disjoint). Prove that this is a matroid (you need to recall the notions of residual graph, and augmenting path in the context of max flow).

2. You are given a set of vectors X in a vector space. Let $(X, \mathcal{I})$ be the set system where a subset S of X is in $\mathcal{I}$ if and only if removing S from X does not change the dimension of X, i.e., $\dim(X) = \dim(X - S)$. Prove that this set system is a matroid. Give an algorithm to check if X has two disjoint subsets S_1, S_2, each of which is a basis of X.

3. Let A be a square $n \times n$ matrix of full rank (i.e., all its rows are linearly independent, and so, all its rows (or columns) are linearly independent). Let R denote the set of rows of A and C denote the set of columns of A. For a subset I of rows R, and a subset J of columns C, $|I| = |J|$, define $A[I, J]$ as the square matrix obtained by considering entries in A which lie in a row corresponding to I and a column corresponding to J, i.e., entries $A_{i,j}$, where $i \in I, j \in J$.

   You are given a subset I of R. Prove that there is a subset J of C, where $|J| = |I|$, such that both the matrices $A[I, J]$ and $A[R - I, C - J]$ are full rank (use matroid intersection).

4. (taken from Sanjeev Arora's course at Princeton) You are given stock price data for 490 different tickers for 1000 consecutive days. Treating the data as off-line data, find out the best constant rebalanced ratio using gradient descent. What is the net worth after 1000 days using this strategy? Now, consider the data as online, and suppose on each day, you can decide on the portfolio distribution before you see the stock prices. Run the online gradient descent algorithm on this data, and report the net worth after 1000 days.

5. A car is driving along a straight line in a plane with constant velocity. There are 3 cell phone towers (call them C_1, C_2, C_3) at locations $(5, 10)$, $(18, 40)$ and $(-6, 90)$. Each of these cell phone towers can tell the distance to the car at a certain point of time (ignore the transmission delay of the signal). Here is the data you see:

Time of Measurement	Tower Number	Distance to Car
1.5	C2	13.5
2.3	C1	30
3.5	C2	6.7
5	C3	55.0
7.1	C1	45.6
9.5	C3	48.9
12	C1	60.1
14	C3	45.1
17	C2	36.9
25	C3	48.9

   You need to find the position of the car at time 0, and its velocity. How will you formulate this as an optimization problem ? Draw the steps taken by the gradient descent algorithm on a 2-D plot, and output the result.