**COL 202**

# TUTORIAL SHEET 4

1. Use mathematical induction to show that when $n$ circles divide the plane into regions, these regions can be colored with two different colors such that no two regions with a common boundary get the same color.

2. Show that if $A, B, C$ are three sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

3. The symmetric difference of two sets $A$ and $B$ is denoted by $A \oplus B$, and defines the set which contains those elements which are either in $A$ or $B$, but not both.
   - Show that $A \oplus B = (A \cup B) - (A \cap B)$
   - Suppose $A, B, C$ are three sets such that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$. Does it follow that $A = B$ ?
   - Suppose $A, B, C, D$ are four sets. Does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$ ?

4. If $f$ and $f \circ g$ are 1-1 functions, does it follow that $g$ is also 1-1 ? If $f$ and $f \circ g$ are onto, does it follow that $g$ is onto ?

5. Show that a set $S$ is infinite if and only if there is a proper subset $A$ of $S$ such that there is a 1-1 correspondence between $A$ and $S$.

6. Pick's theorem says that the area of a simple polygon $P$ in the plane with vertices that are all lattice points (i.e., points with integer coordinates) equals $I(P) + B(P)/2 - 1$, where $I(P)$ and $B(P)$ are the number of lattice points in the interior of $P$ and on the boundary of $P$. Use strong induction on the number of vertices of $P$ to prove this theorem. [ Hint : for the base case, first prove the theorem for rectangles, then right triangles and then for arbitrary triangles by noticing that the area of a triangle is the area of a larger rectangle containing it minus areas of at most three triangles subtracted].

7. Consider the following game: the game begins with $n$ match sticks. Two players take turns removing match sticks one, two or three at a time. The player removing the last match stick loses. Use strong induction to prove that if each player plays the best strategy then the first player wins if the remainder of $n$ when divided by 4 is 0, 2, or 3, and the second player wins otherwise.