COL 202 Aug 22, 17, 18

TUTORIAL SHEET 3

1. Let $a_1, \ldots, a_n$ be positive real numbers. The arithmetic mean of these numbers is defined as $A = (a_1 + a_2 + \ldots + a_n)/n$, and the geometric mean of these numbers is defined as $G = (a_1 a_2 \ldots a_n)^{1/n}$. Use mathematical induction to show that $A \geq G$.

2. Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row such that the average of any two of these numbers never appears between them (Hint: show that it suffices to prove this when n is a power of 2, and use induction in this case).

3. Suppose we want to prove that for all positive integers n,

$$\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

   - Show that if we try to prove this by mathematical induction, the base step works, but we get stuck in the induction step.
   - Show that we can use mathematical induction to prove the stronger inequality:

$$\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}.$$

4. Use the well-ordering principle to show that there is no solution in positive integers to the equation

$$a^2 + b^2 = 3(s^2 + t^2).$$

   Hint: consider all such tuples (a, b, c, d), and choose the one for which a is smallest. Now argue that a and b both must be multiples of 3, and continue.

5. The numbers $1, 2, \ldots, 11$ are written on a board. We repeat the following process till only one number remains: pick any two numbers, and replace them by the absolute value of their difference. Prove that we cannot end up with 4.

6. Show that a $6 \times n$ board ($n \geq 2$) can be tiled with L-shaped tiles, without gap and overlapping. Each L-shaped tiles cover three squares.

7. Which of the powers of 9 ($9^0, 9^1, 9^2, \ldots$) have 9 as a unit digit ? Prove by induction.