COL 202

TUTORIAL SHEET 13

1. How many spanning trees are there in an n-wheel graph W_n (a graph with n "outer" vertices in a cycle, each connected to an $(n+1)$st "hub" vertex), when $n \geq 3$?

   Solution Hint: It is easier to first look at the "fan" graph, which is a graph with n "outer" vertices on a path, each connected to an $(n+1)$st "hub" vertex. Let F_n denote the number of spanning trees of such a graph. Label the vertices on the path as $1, 2, \ldots, n$. Consider a spanning tree – if we remove the edges incident with the hub vertex, the tree breaks into several simple paths. Let k denote the number of vertices on such a path containing vertex "1". Then it is easy to check that if we only look at those spanning trees, where the number of vertices in the corresponding path containing "1" is k, then the number of such trees would be $k \cdot F_{n-k}$ because the vertices numbered $1, \ldots, k$ will be in one path in the spanning tree and there are k choices about how to connect this path to the hub vertex. For the remaining vertices, we just need to find a spanning tree over a fan on $n-k$ vertices. Therefore, we get the recurrence
$$F_n = \sum_{k=1}^{n} k \cdot F_{n-k},$$
   where F_0 is 1. From this expression, deduce that
$$F_n = 3F_{n-1} - F_{n-2}.$$

   The case for wheel graph is similar, though slightly trickier. Let T_n denote the number of spanning trees, and label the vertices $1, 2, \ldots, n$ in a circular order. Again suppose we remove the edges incident to the hub vertex, and look at the path containing vertex 1. Suppose this path has k vertices. Then, there are k possibilities for the set of vertices in this path: they could be $1, 2, \ldots, k$ or $n, 1, 2, \ldots, k-1$, or $n-1, n, 1, 2, \ldots, k-2$ and so on. For each of these choices, we have k choices for how to add the edge from the hub vertex to this path. Also, note that if we remove this path from the graph, the remaining graph is a fan graph on $n-k$ vertices. Therefore, we get
$$T_n = \sum_{k=1}^{n} k^2 \cdot F_{n-k}.$$

   From this deduce that
$$T_n = T_{n-1} + F_n + F_{n-1}.$$

   Now use generating functions.

2. Prove that in every tree, any two paths with maximum length have a node in common.

3. Prove that either a graph G or its complement G (includes only those edges which are not in G) is connected.

4. A sequence $d_1, d_2, ..., d_n$ is called graphic if it is the degree sequence of a simple graph.

   (a) Show that there is a simple graph with vertices $v_1, v_2, ..., v_n$ such that $\deg(v_i) = d_i$ for $i = 1, 2, ..., n$ and v_1 is adjacent to $v_2, ..., v_{d_i+1}$.

   (b) Show that a sequence $d_1, d_2, ..., d_n$ of nonnegative integers in non-increasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_n$ so that the terms are in nonincreasing order is a graphic sequence.

5. Fleury's algorithm, constructs Euler circuits by first choosing an arbitrary vertex of a connected multigraph, and then forming a circuit by choosing edges successively. Once an edge is chosen, it is removed. Edges are chosen successively so that each edge begins where the last edge ends, and so that this edge is not a cut edge unless there is no alternative. Prove that Fleury's algorithm always produces an Euler circuit.

   Solution Hint: Note that an edge is called a cut edge if its removal increases the number of connected components in the graph. First prove the following useful fact: consider a graph (which may not be connected) where every vertex has even degree. Then it does not have a cut edge.

   Now consider such a path P starting with a vertex s constructed by the algorithm. First observe that it must end at s (why?). The only worry is that it may not contain all the vertices of the graph. So suppose this is the case. Let V_1 be the vertices in P and V_2 be the vertices not covered by P. Let v be the last vertex on P such that there is an edge between v and a vertex in V_2 (such a vertex v must exist because the graph is connected). Let the vertex after v in the path P be w. Also let (v, x) be an edge where $x \in V_2$. First observe that when we reach v during the algorithm, we have deleted all the edges in P so far. We claim that (v, w) is a cut-edge. Indeed, after remove this edge, we disconnect x from rest of P (recall the definition of v). Still our algorithm took this edge. It must be the case that all edges incident to v are cut edges. But now we claim that (v, x) is not a cut edge. Indeed, if we remove all of P, the remaining graph has even degree for every vertex. So, (v, x) cannot be a cut edge for this graph. And so, it would not be a cut edge when the algorithm comes to v (why?).

6. The distance between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The radius of a graph is the minimum over all vertices v of the maximum distance from v to another vertex. The diameter of a graph is the maximum distance between two distinct vertices.

   - Show that if the diameter of the simple graph G is at least four, then the diameter of its complement G is no more than two.

     Solution Hint: Let u, v be two vertices in G such that distance between u and v is 4. Now consider any two vertices x and y. We need to show that the distance between them is at most 2 in the complement graph. Assume that (x, y) is an edge in G – otherwise there will be a path of length 1 between them in the complement graph. Now observe that each vertex in G is joined to either x or y (or both) by an edge – indeed, otherwise there will be a path of length 2 between x and y (via this vertex) in the complement graph. So assume that (u, x) is an edge. If (v, x) is also an edge, then there is a path of length 2 from x to y in G, which is not possible. So, (v, y) must be an edge in G. But then u, x, y, v is a path of length 3 in G, a contradiction.

- Show that if the diameter of the simple graph G is at least three, then the diameter of its complement G is no more than three.

  Solution Hint: Suppose u and v are two vertices such that the distance between them is 3 in G. As above, let x and y be any two vertices. As argued above, we can assume that (u, x) and (v, y) are edges in G. It follows that (u, y) and (v, x) are NOT in G – otherwise we have a path of length 2 between u and v. Now, note that the complement graph has the following edges: $(x, v), (v, u), (u, y)$. Thus, there is a path of length 3 between u and v in the complement graph.